Published by Institute of Physics Publishing for SISSA
Received: *November 29, 2005*
Accepted: *December 14, 2005*
Published: *January 9, 2006*

Topological strings and large N phase transitions I: Nonchiral expansion of q-deformed Yang-Mills theory

Nicola Caporaso,[a] Michele Cirafici,[b] Luca Griguolo,[c] Sara Pasquetti,[c] Domenico Seminara[a] and Richard J. Szabo[b]

[a] *Dipartimento di Fisica, Polo Scientifico Università di Firenze, INFN Sezione di Firenze Via G. Sansone 1, 50019 Sesto Fiorentino, Italy*
[b] *Department of Mathematics, Heriot-Watt University Colin Maclaurin Building, Riccarton, Edinburgh EH14 4AS, U.K.*
[c] *Dipartimento di Fisica, Università di Parma, INFN Gruppo Collegato di Parma Parco Area delle Scienze 7/A, 43100 Parma, Italy*
E-mail: caporaso@fi.infn.it, M.Cirafici@ma.hw.ac.uk, griguolo@fis.unipr.it, pasquetti@fis.unipr.it, seminara@fi.infn.it, R.J.Szabo@ma.hw.ac.uk

Abstract: We examine the problem of counting bound states of BPS black holes on local Calabi-Yau threefolds which are fibrations over a Riemann surface by computing the partition function of q-deformed Yang-Mills theory on the Riemann surface. We study in detail the genus zero case and obtain, at finite N, the instanton expansion of the gauge theory. It can be written exactly as the partition function for $\mathrm{U}(N)$ Chern-Simons gauge theory on a Lens space, summed over all non-trivial vacua, plus a tower of non-perturbative instanton contributions. The correspondence between two and three dimensional gauge theories is elucidated by an explicit mapping between two-dimensional Yang-Mills instantons and flat connections on the Lens space. In the large N limit we find a peculiar phase structure in the model. At weak string coupling the theory reduces exactly to the trivial flat connection sector with instanton contributions exponentially suppressed, and the topological string partition function on the resolved conifold is reproduced in this regime. At a certain critical point all non-trivial vacua contribute, instantons are enhanced and the theory appears to undergo a phase transition into a strong coupling regime. We rederive these results by performing a saddle-point approximation to the exact partition function. We obtain a q-deformed version of the Douglas-Kazakov equation for two-dimensional Yang-Mills theory on the sphere, whose one-cut solution below the transition point reproduces the resolved conifold geometry. Above the critical point we propose a two-cut solution that should reproduce the chiral-antichiral dynamics found for black holes on the Calabi-Yau threefold and the Gross-Taylor string in the undeformed limit. The transition from the strong coupling phase to the weak coupling phase appears to be of third order.

Keywords: Topological Strings, Field Theories in Lower Dimensions, Nonperturbative Effects, Brane Dynamics in Gauge Theories.

Contents

JHEP01(2006)035

1. Introduction

Topological string amplitudes have been recently connected to the counting of microstates of four-dimensional BPS black holes in a novel and highly non-trivial way [1]. This proposal extends and generalizes a series of beautiful results [2 – 5] concerning the entropy of BPS black holes arising in compactifications of Type II superstrings on Calabi-Yau threefolds. A precise relation between the mixed ensemble black hole partition function $Z_{\rm BH}$ and the topological string vacuum amplitude $Z_{\rm top}$ has been conjectured as $Z_{\rm BH}=|Z_{\rm top}|^2$. In particular, it has been suggested that for a large black hole charge N the relation should be valid at any order in the $\frac{1}{N}$ expansion, taking into account the *perturbative* definition of $Z_{\rm top}$. The proposal of [1] actually goes further, pushing forward the possibility that *nonperturbative* topological string amplitudes, including $O(\,{\rm e}^{\,-1/g_s})$ corrections, could be defined from the conjectured relation.

In order to check this proposal, one should find some Calabi-Yau backgrounds in which both sides of the relation can be computed independently. For this task a general class of suitable non-compact Calabi-Yau threefolds has been recently studied [6], generalizing the original example presented in [7]. The relevant threefold X is the total space of a rank 2 holomorphic vector bundle over a genus g Riemann surface Σ_g,

$$X = \mathcal{O}(p+2g-2)\oplus\mathcal{O}(-p) \ \longrightarrow\ \Sigma_g\,, \tag{1.1}$$

where $\mathcal{O}(m)$ is a holomorphic line bundle of degree m over Σ_g (in [7] the case $g=1$ was considered). The counting of BPS states on these geometries has been claimed to reduce to computing the partition function of a peculiar deformation of Yang-Mills theory on Σ_g called q-deformed Yang-Mills theory. Starting from this result one should ask if the relation with the perturbative topological string amplitudes holds in this case. Happily the partition function $Z_{\rm top}$ for these geometries has been computed very recently [8] and the consistency check amounts to reproducing these amplitudes as the large N limit of q-deformed Yang-Mills theory on Σ_g. In [6] a large N expansion has been performed and the conjecture was confirmed, but with a couple of important subtleties. Firstly, one should include in the definition of $Z_{\rm top}$ a sum over a U(1) degree of freedom identified with a Ramond-Ramond flux through the Riemann surface. Secondly, and more importantly, the relevant topological string partition function implies the presence of $|2g-2|$ stacks of D-branes inserted in the fibers of X. An explanation of this unexpected feature in terms of extra closed string moduli, related to the non-compactness of X, has been offered in [9]. Recent works on the conjecture include [10]–[20].

A central role in all these advances has been played by the q-deformed U(N) Yang-Mills theory. At finite N it should reproduce the counting of BPS states of a black hole that arises from N D4-branes wrapping the submanifold $C_4=\mathcal{O}(-p)\to\Sigma_g$ of X with any number of D0-branes and D2-branes wrapping Σ_g. This statement is equivalent to the surprising result that the partition function of q-deformed Yang-Mills theory is the generating functional of the Euler characteristics of the moduli spaces of U(N) instantons in the topologically twisted $\mathcal{N}=4$ Yang-Mills theory on C_4 [6]. It therefore provides instanton counting on very non-trivial non-compact four-manifolds. On the other hand, its large N

limit is described by the topological A-model on X, a closed topological string theory with six-dimensional target space. That two-dimensional Yang-Mills theory should be related to a string theory in the large N limit is not entirely unexpected due to the well-known Gross-Taylor expansion [21]–[23]. At large N, the partition function of two-dimensional Yang-Mills theory on a Riemann surface Σ_g almost factorizes into two copies (called chiral and antichiral) of the same theory of unfolded branched covering maps, the target space being Σ_g itself. The chiral-antichiral factorization is violated by some geometrical structures called orientation-reversing tubes.

The emergence of chiral and antichiral sectors was also observed in [6] in studying the q-deformed version of two dimensional Yang-Mills theory and it implies the appearance of the modulus squared $|Z_{\text{top}}|^2$, a crucial ingredient in checking the relation with black hole physics. Trying to understand the relation between Gross-Taylor string theory and the topological string theory underlying q-deformed Yang-Mills theory at large N is quite tempting. Moreover, due to the intimate relation with four-dimensional gauge theories, it is important to understand better how topological strings emerge from two-dimensional gauge degrees of freedom and whether or not the string description, with its chiral-antichiral behaviour, is restricted to a limited region of parameter space. It is well-known that the familiar Yang-Mills theory on the sphere S^2 undergoes a large N phase transition at a particular value of the coupling constant [24]. A strong coupling phase, wherein the theory admits the Gross-Taylor string description, is separated by a weak-coupling phase with gaussian field theoretical behaviour. Instanton configurations induce the transition to strong coupling [25], while the entropy of branch points appears to be responsible for the divergence of the string expansion above the critical point [26, 27].

In this paper we will study in detail q-deformed Yang-Mills theory on S^2 and its relation with topological string theory on the threefold $X = \mathcal{O}(p-2) \oplus \mathcal{O}(-p) \to \mathbb{P}^1$. We will show that the theory still exhibits a phase transition at large N. A companion paper [28] will be devoted to a precise comparison with topological strings on X, showing how this description arises in the strong coupling phase. The present paper focuses on the gauge theoretical aspects and on the determination of the phase transition. We study the weak-coupling phase that is still represented by a topological string but without chiral-antichiral factorization. We also concentrate on a detailed description of the strong-coupling regime, discussing both conceptual and technical aspects.

The structure of the paper is as follows. In section 2 we review the relation between black hole entropy, topological string theory on X and q-deformed Yang-Mills theory. This section is intended to be a brief introduction to the background of the subject which may be skipped by the experts. We describe how the string coupling g_s produces the deformation and how the integer p is related to the effective area seen by the string theory. The expectations of the large N limit are also discussed. In section 3 we discuss the gauge theoretical structure of the theory. In section 3.1 we describe the general properties of the q-deformed theory and point out how in an appropriate double scaling limit, $p \to \infty$, $g_s \to 0$ with $g_s p$ fixed, the usual Yang-Mills theory is recovered. In section 3.2 we obtain, at finite N, the instanton expansion of the gauge theory. In this form the comparison with the $\mathcal{N} = 4$ topologically twisted theory should be easier and we make some comments about it. In section 3.3 we show that the partition function can be written exactly as the partition function of $\mathrm{U}(N)$ Chern-Simons theory on the Lens space $L_p = S^3/\mathbb{Z}_p$, summed over all non-trivial vacua, plus a tower of non-perturbative instanton contributions. The relation between q-deformed Yang-Mills theory and Chern-Simons theory on Seifert manifolds [29] is thereby unveiled in this case. We make this relation explicit by explicitly constructing the mapping between instantons on S^2 and flat connections on L_p. We also discuss the possible description in terms of open topological strings, suggested by the presence of the underlying Chern-Simons theory. In section 4 we observe a peculiar behaviour dependent on the integer p specifying the Calabi-Yau threefold and the string coupling constant g_s. In the large N limit, for all $p \geq 1$ and $g_s N < p \log(\sec(\frac{\pi}{p})^2)$, the theory completely reduces to the trivial flat connection sector and all instanton contributions are exponentially suppressed. This behaviour is reminescent of the weak-coupling phase appearing in the undeformed case at small area. Not surprisingly, due to the relation with Chern-Simons theory on L_p, the topological string partition function on the resolved conifold is reproduced in this regime. Instead, for $p > 2$ and $g_s N > p \log(\sec(\frac{\pi}{p})^2)$ all non-trivial vacua contribute, instantons are enhanced and the theory undergoes a phase transition to a strong coupling regime. These results are obtained by expliciting evaluating the ratio between the one-instanton and the zero-instanton contributions to the partition function, in the spirit of [25]. In section 5 we arrive at the same conclusions by performing a saddle point approximation to the exact partition function at large N. In section 5.1 a deformed version of the Douglas-Kazakov equation is derived and is related to the undeformed one as $p \to \infty$. In section 5.2 we show that its one-cut solution, valid below the transition point, reproduces the resolved conifold geometry found in the weak-coupling phase. Section 6 contains the most important results: we study the theory above the critical value. In section 6.1 we propose an exact two-cut solution of the saddle-point equation, that should reproduce the chiral-antichiral dynamics found for black-holes on X and the Gross-Taylor string for $p \to \infty$. The equations for the end-points of the cuts are rather complicated, involving elliptic functions of the third type. Nevertheless they can be written in an elegant way, reducing to the Douglas-Kazakov equations in the $p \to \infty$ limit (section 6.2). We show that for $p \leq 2$ they do not admit solutions, while above the critical value, for $p > 2$, they always do uniquely. The transition curve is then recovered in section 6.3. A third-order phase transition, generalizing non-trivially the Douglas-Kazakov result to the q-deformed case, is finally found in section 6.4, by using an expansion in modular functions around the critical point. We also give evidences of the arising of the topological string expansion at large 't Hooft parameters in section 6.5, by using the modular properties of the exact solution. In section 7 we draw our conclusions and speculate on possible applications of these results to black hole physics. Three appendices at the end of the paper contain technical details of the computations presented in the main text.

Note added: as this manuscript was being completed, refs. [77] and [78] appeared, presenting an overlap with the results of this paper.

JHEP01(2006)035

2. Black holes, topological strings and q-deformed Yang-Mills theory

We start by reviewing the conjecture presented in [1]. Consider Type II superstring theory on $X \times \mathbb{R}^{3,1}$, where X is a Calabi-Yau threefold. A BPS black hole can be obtained, in this context, by wrapping D6, D4, D2 and D0 branes around holomorphic cycles in X. The charges carried by the black holes are determined by appropriately choosing the holomorphic cycles. Usually D6 and D4-brane charges are referred to as "magnetic" while D2 and D0-brane charges are "electric", with the intersection pairings in X giving rise to electric-magnetic duality in four dimensional space. One can define a partition function for a mixed ensemble of BPS black hole states by fixing the magnetic charges Q_6 and Q_4 and summing over the D2 and D0 charges with fixed chemical potentials ϕ_2 and ϕ_0 to get

$$Z_{\mathrm{BH}}(Q_6, Q_4, \phi_2, \phi_0) = \sum_{Q_2, Q_0} \Omega(Q_6, Q_4, Q_2, Q_0)\ \exp\big[-Q_2\phi_2 - Q_0\phi_0\big] , \tag{2.1}$$

where $\Omega(Q_6, Q_4, Q_2, Q_0)$ is the contribution from BPS states with fixed D-brane charges. The conjecture relates the partition function (2.1) to the topological string vacuum amplitude as

$$Z_{\mathrm{BH}}(Q_6, Q_4, \phi_2, \phi_0) = \big|Z_{\mathrm{top}}(g_s, t_s)\big|^2 , \tag{2.2}$$

where $Z_{\mathrm{top}}(g_s, t_s)$ is the A-model topological string partition function with the identifications

$$g_s = \frac{4\pi\,\mathrm{i}}{\frac{\mathrm{i}}{\pi}\,\phi_0 + Q_6} , \qquad t_s = \frac{1}{2}\, g_s\left(-\frac{\mathrm{i}}{\pi}\,\phi_2 + N Q_4\right) \tag{2.3}$$

for the topological string coupling g_s and the Kähler modulus t_s. For recent reviews on topological strings, see [30, 31]

This proposal can be considered as an all orders generalization of some well-known properties of BPS black holes in $\mathcal{N} = 2$ supergravity [2–5]. Black hole solutions are found in the background of $2n_v + 2$ gauge fields ($n_v + 1$ magnetic duals of the others) and they carry charges (P^I, Q_I) with respect to them. The gauge fields are organized into n_v vector multiplets plus one graviphoton (arising from the supergravity multiplet). The scalar fields of the theory X^I, that may be regarded as position dependent moduli of the Calabi-Yau threefold on which superstrings have been compactified, have fixed values at the black hole horizon determined only by the charges. This phenomenon is called the attractor mechanism [32, 33]. In turn this relation can be expressed in purely geometric terms by using the periods of the holomorphic three-form Ω along the cycles A_I, B^I of the Calabi-Yau threefold at the horizon as

$$P^I = \mathrm{Re}(X^I) = \oint_{A_I} \mathrm{Re}(\Omega) , \quad Q_I = \mathrm{Re}(F_I) = \oint_{B^I} \mathrm{Re}(\Omega) , \tag{2.4}$$

where $F_I = \frac{\partial F_0}{\partial X^I}$ and F_0 is the prepotential of X. The black hole entropy S_{BH} is a function only of the charges in the extremal case. By means of the explicit expressions in eq. (2.4) one obtains

$$S_{\mathrm{BH}} \simeq Q_I X^I - P^I F_I . \tag{2.5}$$

The entropy thus appears as a Legendre transform of the prepotential F_0, which is also known to be the genus zero free energy of topological strings in the Calabi-Yau background. At the supergravity level it is possible to include higher-derivative corrections proportional to $R^2 T^{2g-2}$, where T is the graviphoton field-strength, and to compute the black hole solutions and their entropies [2–5]. At the first non-trivial order ($g = 1$) the relation with the topological string free energy still holds when one includes quantum corrections to the prepotential coming from one-loop amplitudes [34, 35]. The conjecture of [1] is consistent with these results and cleverly generalizes them at all orders in the perturbative topological string expansion. However, the proposal is even somewhat more startling. Because the partition function $Z_{\rm BH}$ makes sense also at the nonperturbative level, eq. (2.1) should provide a nonperturbative definition of the topological string partition function. In particular the presence of a square-modulus signals a breaking of holomorphicity at the nonperturbative level. We will come back to this point later on.

It is of course natural to attempt to check this conjecture in some explicit examples. One requires a Calabi-Yau threefold X on which to engineer a BPS black hole whose partition function could be computed by a counting of microstates, while at the same time being simple enough to enable the computation of the topological string partition function to all orders. While for compact manifolds the task seems out of reach presently, in the non-compact case there is the general class of threefolds (1.1) on which the problem has been attacked [6, 7]. The study of topological strings on these backgrounds and the related counting of microstates have also produced a number interesting independent results. As we will see, different gauge theories in diverse dimensions appear to be related by their common gravitational ancestor.

2.1 Counting microstates in $\mathcal{N} = 4$ and q-deformed Yang-Mills theories

Let us begin by describing the counting of microstates. It consists of counting bound states of D4, D2 and D0-branes, where the D4 branes wrap the four cycle C_4 which is the total space of the holomorphic line bundle

$$C_4 = \mathcal{O}(-p) \longrightarrow \Sigma_g \tag{2.6}$$

and the D2-branes wrap the Riemann surface Σ_g. The number of D4-branes is fixed to be N and one should count in the ensemble of bound states on it. The natural way of doing the computation is by studying the relevant gauge theory on the brane, where the presence of chemical potentials for D2 and D0-branes corresponds to the introduction of some interactions. According to the general framework [36] the worldvolume gauge theory on the N D4-branes is the $\mathcal{N} = 4$ topologically twisted $\mathrm{U}(N)$ Yang-Mills theory on C_4. The presence of chemical potentials is simulated by turning on the observables in the theory given by

$$S_c = \frac{1}{2g_s} \int_{C_4} \mathrm{Tr}\big(F \wedge F\big) + \frac{\theta}{g_s} \int_{C_4} \mathrm{Tr}\big(F \wedge K\big)\,, \tag{2.7}$$

where F is the Yang-Mills field strength and K is the unit volume form of Σ_g. The relation between the gauge parameters g_s, θ and the chemical potentials ϕ_0, ϕ_2 is

$$\phi_0 = \frac{4\pi^2}{g_s}\,, \quad \phi_2 = \frac{2\pi\,\theta}{g_s}\,, \tag{2.8}$$

in accordance with the identifications for the D0 and D2-brane charges q_0, q_2 as

$$q_0 = \frac{1}{8\pi^2}\int_{C_4} \mathrm{Tr}\big(F\wedge F\big)\,, \quad q_2 = \frac{1}{2\pi}\int_{C_4} \mathrm{Tr}\big(F\wedge K\big)\,. \tag{2.9}$$

Obtaining Z_{BH} is therefore equivalent to computing the expectation value in topologically twisted $\mathcal{N}=4$ Yang-Mills theory given by

$$Z_{\mathrm{BH}} = \Big\langle \exp\Big[-\frac{1}{2g_s}\int_{C_4}\mathrm{Tr}\big(F\wedge F\big) - \frac{\theta}{g_s}\int_{C_4}\mathrm{Tr}\big(F\wedge K\big)\Big]\Big\rangle = Z_{\mathcal{N}=4}\,. \tag{2.10}$$

The general structure of this functional integral has been explored in [37]. There it was shown that with an appropriate gauge fixing the partition function $Z_{\mathcal{N}=4}$ has an expansion of the form

$$Z_{\mathcal{N}=4} = \sum_{q_0,q_2} \Omega(q_0,q_2;N)\ \exp\Big(-\frac{4\pi^2}{g_s}\,q_0 - \frac{2\pi\,\theta}{g_s}\,q_2\Big)\,, \tag{2.11}$$

where $\Omega(q_0,q_2;N)$ is (under suitable assumptions) the Euler characteristic of the moduli space of $\mathrm{U}(N)$ instantons on C_4 in the topological sector labelled by the zeroth and second Chern numbers q_0 and q_2. We see that the counting of microstates is equivalent to an instanton counting in the $\mathcal{N}=4$ topological gauge theory. This is of course still a formidable problem, because no general strategy exists in the case of non-compact manifolds and very few results [38–40] have been obtained in this context. Moreover, we expect that C_4 has a very complicated instanton moduli space, especially in the higher genus case.

A key observation [7] allows one to reduce the computation to a two-dimensional problem. The $\mathcal{N}=4$ topological gauge theory is believed to be invariant under certain massive deformations which drastically simplify the theory. By using a further deformation which corresponds to a $\mathrm{U}(1)$ rotation on $\mathcal{O}(-p)$, it was argued in [7] that the theory localizes to $\mathrm{U}(1)$-invariant modes and reduces to an effective gauge theory on Σ_g. One could expect that the gauge theory is still fully topological from the two-dimensional point of view. In [7] it was shown that the non-triviality of the fibration $\mathcal{O}(-p)$ generates an extra term in the effective two-dimensional action which basically contains all the information about the four-dimensional structure. It is given by

$$S_p = -\frac{p}{2g_s}\int_{\Sigma_g}\mathrm{Tr}\,\Phi^2\,K\,, \tag{2.12}$$

where

$$\Phi(z) = \oint_{S^1_{z,|u|=\infty}} A \tag{2.13}$$

parameterizes the holonomy of the gauge field A around a circle at infinity in the fiber over the point $z\in\Sigma_g$, with u a local complex coordinate of $\mathcal{O}(-p)$. The relevant two-dimensional action becomes

$$S_{\mathrm{YM}_2} = \frac{1}{g_s}\int_{\Sigma_g}\mathrm{Tr}\big(\Phi\,F\big) + \frac{\theta}{g_s}\int_{\Sigma_g}\mathrm{Tr}\,\Phi\ K - \frac{p}{2g_s}\int_{\Sigma_g}\mathrm{Tr}\,\Phi^2 K. \tag{2.14}$$

This is just the action of two-dimensional Yang-Mills theory on the Riemann surface Σ_g. In the case that Σ_g is the torus T^2 ($g = 1$), ordinary two-dimensional Yang-Mills theory provides the correct instanton counting on $\mathcal{O}(-p) \to T^2$ [7].

However, in the general case there is an important subtlety. The two-dimensional path-integral should take into account the new degree of freedom Φ, which is periodic due to its origin as the holonomy of the gauge field at infinity. It can be shown [6] that periodicity affects the path integral measure in a well-defined way. As a consequence, the emerging theory has a natural interpretation as a peculiar q-deformation of two-dimensional Yang-Mills theory. This deformation has already been introduced in [41] with different motivations. The exact form of the partition function obtained by performing the path integral with the action (2.14) and periodic measure can be derived [6] in a spirit similar to [42] or in a combinatorial approach resembling that of [43]. The final result is

$$Z_{\mathcal{N}=4} = Z^q_{\mathrm{YM}} = \sum_R \dim_q(R)^{2-2g} \; q^{\frac{p}{2}\, C_2(R)} \;\; \mathrm{e}^{\,\mathrm{i}\,\theta\, C_1(R)} \tag{2.15}$$

This is to be compared with the partition function of ordinary Yang-Mills theory on Σ_g given by the Migdal expansion [44]

$$Z_{\mathrm{YM}} = \sum_R \dim(R)^{2-2g} \;\; \mathrm{e}^{\,\frac{g^2 A}{2}\, C_2(R)} \;\; \mathrm{e}^{\,\mathrm{i}\,\theta\, C_1(R)} \,, \tag{2.16}$$

where R runs through the unitary irreducible representations of the gauge group $\mathrm{U}(N)$, $\dim(R)$ is its dimension, and $C_1(R)$ and $C_2(R)$ are respectively its first and second Casimir invariants. The dimensionless combination g^2A of the Yang-Mills coupling constant g^2 and area A of Σ_g is the effective coupling of the theory, as dictated by invariance under area-preserving diffeomorphisms. With R_i labelling the lengths of the rows in the Young tableau corresponding to the irreducible representation R, the main effect of the deformation is to turn the ordinary dimension of the R given by

$$\dim(R) = \prod_{1\le i<j\le N} \frac{R_i - R_j + j - i}{j-i} \tag{2.17}$$

into the *quantum* dimension

$$\dim_q(R) = \prod_{1\le i<j\le N} \frac{\big[R_i - R_j + j - i\big]_q}{\big[j-i\big]_q} = \prod_{1\le i<j\le N} \frac{\big[q^{(R_i-R_j+j-i)/2} - q^{-(R_i-R_j+j-i)/2}\big]}{\big[q^{(j-i)/2} - q^{-(j-i)/2}\big]} \,, \tag{2.18}$$

where the deformation parameter q is related to the coupling g_s through

$$q = \mathrm{e}^{\,-g_s} \,. \tag{2.19}$$

Clearly as $g_s \to 0$ the quantum dimension goes smoothly into the classical one. The effective dimensionless coupling of the gauge theory is $g^2A = g_s p$. The deformation arises only for $g \neq 1$ and the conclusions of [7] still hold.

Eq. (2.15) provides the solution of the difficult problem of instanton counting, as it represents the answer on all four-manifolds C_4. The direct comparison with eq. (2.11) seems at first sight puzzling, because the expansion of the q-deformed gauge theory is in e^{-g_s} and not in e^{-1/g_s}. This implies that a modular transformation is required in eq. (2.15) and we will discuss this point in section 3.

2.2 Large N limit and topological strings

Let us come back now to the conjecture. Eq. (2.15) gives, in principle, the exact expression for the black hole partition function Z_{BH}. In order to establish the connection with the perturbative topological string partition function on X, we need to take the limit of large charges and consider the large N limit of the q-deformed Yang-Mills theory. It is important, first of all, to set the relation between the parameters of the gauge theory describing the black hole and the data of the closed topological string theory. According to the conjecture, the moduli of the Calabi-Yau manifold are fixed by the black hole attractor mechanism. The real parts of the projective coordinates $(X^0,\ X^1)$ on Calabi-Yau moduli space are connected to the magnetic charges (D6 and D4-branes) while their imaginary parts are the chemical potentials ϕ_0 and ϕ_2 for the D0 and D2-brane charges. Here D6-branes are absent and we have N D4-branes. Their charges should be measured in terms of electric units of D2-branes wrapping Σ_g. By evaluating the intersection number of Σ_g and the four-cycle C_4 on which the D4-branes are wrapped, it can be shown that the relevant charge is $p+2g-2$. The projective moduli for the closed topological string are therefore

$$X^0 = \mathrm{i}\,\frac{\phi_0}{\pi}\,, \quad X^1 = (p+2g-2)N - \mathrm{i}\,\frac{\phi_2}{\pi}\,, \tag{2.20}$$

which from eq. (2.8) implies

$$X^0 = \frac{4\pi\,\mathrm{i}}{g_s}\,, \quad X^1 = (p+2g-2)N - 2\,\mathrm{i}\,\frac{\theta}{g_s}\,. \tag{2.21}$$

The Kähler modulus t_s of the base Σ_g is given by $t_s = 2\pi\,\mathrm{i}\,X^1/X^0$. The closed topological string emerging from the large N limit is thereby expected to possess the modulus

$$t_s = (p+2g-2)\,\frac{N g_s}{2} - \mathrm{i}\,\theta\,. \tag{2.22}$$

This is the first very non-trivial feature that should be reproduced by the large N limit.

The second non-trivial point is that a square modulus structure should emerge. This is not completely unexpected due to the relation with the large N limit of ordinary Yang-Mills theory. In the Gross-Taylor expansion [23] two types of representations contribute to the limit, called the chiral representations (with much less than N Young tableaux boxes) and the antichiral representations (with order N boxes). The partition function is almost factorized into two copies, apart from the contribution of some geometrical structures called orientation-reversing tubes that are required to complete the description. According to the same logic, one could wonder whether Z^q_{YM} factorizes as well. The choice of relevant representations is dictated by the Casimir dependence of the partition function, which is unchanged by the deformation. We expect

$$Z^q_{\mathrm{YM}} \simeq Z^{q,+}_{\mathrm{YM}}\, Z^{q,-}_{\mathrm{YM}}\,. \tag{2.23}$$

Moreover, one would expect that the chiral q-deformed Yang-Mills partition function $Z^{q,+}_{\rm YM}$ could be written as a holomorphic function of t_s and identified with the topological string amplitude $Z_{\rm top}(g_s, t_s)$ on X. In [6] these expectations have been confirmed, but with some important subtleties.

For genus $g > 1$ one finds [6]

$$Z^q_{\rm YM}(\Sigma_g) = \sum_{l=-\infty}^{\infty} \sum_{\hat R_1,\dots,\hat R_{2g-2}} Z^{q{\rm YM},+}_{\hat R_1,\dots,\hat R_{2g-2}}(t_s + p\, g_s l)\, Z^{q{\rm YM},-}_{\hat R_1,\dots,\hat R_{2g-2}}(\,\bar t_s - p\, g_s l)\,, \tag{2.24}$$

where $\hat R_i$ are irreducible representations of $\mathrm{SU}(N)$ and the chiral block $Z^{q{\rm YM},+}_{\hat R_1,\dots,\hat R_{2g-2}}(t_s)$ agrees exactly with the perturbative topological string amplitude on X [8] with $2g-2$ stacks of D-branes inserted in the fiber. It depends explicitly on the choice of $2g-2$ arbitrary Young tableaux which correspond to the boundary degrees of freedom of the fiber D-branes. When all the Young tableaux are taken to be trivial, one recovers the expected closed topological string partition function. The chiral and anti-chiral parts are sewn along the D-branes and summed over them. The extra integer l which shifts the Kähler modulus is not the naive expectation which would take only the $l = 0$ term and neglect the fiber D-brane contributions.

For $g = 0$, the case studied extensively in this paper, the result is slightly different and given by

$$Z^q_{\rm YM}(S^2) = \sum_{l=-\infty}^{\infty} \sum_{\hat R_1,\, \hat R_2} Z^{q{\rm YM},+}_{\hat R_1,\, \hat R_2}(t_s + p\, g_s l)\, Z^{q{\rm YM},-}_{\hat R_1,\, \hat R_2}(\,\bar t_s - p\, g_s l) \tag{2.25}$$

with

$$Z^{q{\rm YM},-}_{\hat R_1,\, \hat R_2}(\,\bar t_s\,) = (-1)^{|\hat R_1|+|\hat R_2|}\, Z^{q{\rm YM},+}_{\hat R_1^\top,\, \hat R_2^\top}(\,\bar t_s\,)\,, \tag{2.26}$$

where $|\hat R|$ is the total number of boxes of the Young tableau of the representation $\hat R$. The genus zero case is also special because it admits a standard description in terms of toric geometry. The fibration $X = \mathcal{O}(p-2) \oplus \mathcal{O}(-p) \to \mathbb{P}^1$ is in fact a toric manifold [28] and the partition function can be written in terms of the topological vertex $C_{\hat R_1 \hat R_2 \hat R_3}(q)$ [45] as

$$Z^{q{\rm YM},+}_{\hat R_1,\, \hat R_2}(t_s) = Z_0\, q^{k_{\hat R_1}/2}\ \mathrm{e}^{-\frac{t_s(|\hat R_1|+|\hat R_2|)}{p-2}} \sum_{\hat R} \mathrm{e}^{-t_s|\hat R|}\, q^{(p-1)k_{\hat R_1}/2}\, C_{0\hat R_1 \hat R^\top}(q)\, C_{0\hat R \hat R_2}(q)\,, \tag{2.27}$$

where $k_{\hat R}$ is related to the Young tableaux labels through $k_{\hat R} = \sum_i \hat R_i(\hat R_i - 2\,i + 1)$ and Z_0 represents the contribution from constant maps. This is the partition function of the topological A-model on X with non-compact D-branes inserted at two of the four lines in the web diagram. The D-branes are placed at a well-defined "distance" $t_s/(p-2)$ from the Riemann surface, thereby introducing another geometrical parameter.

We thus observe an apparent discrepancy between the prediction of [1] that $Z_{\rm BH} = |Z_{\rm top}|^2$ and the explicit computation leading to

$$Z_{\rm BH} = \sum_{b,l} \big|Z^{(b,l)}_{\rm top}\big|^2\,. \tag{2.28}$$

We have indicated with the index b the sum over chiral blocks with branes inserted. The extra sum over the integer l originates from the U(1) degrees of freedom contained in the original gauge group U(N), which has been interpreted as a sum over Ramond-Ramond fluxes through the Riemann surface [7]. The sum over the fiber D-branes seems instead related to the fact that the Calabi-Yau is non-compact and has more moduli coming from the non-compact directions [9]. We will not enter into this subtle aspect of the comparison except for noticing that, at least for $g = 0$, the sum over the "external" branes in the complete partition function enters on the same footing as the sum over the topological string amplitude constituents. This "external" sum is weighted with a different Kähler parameter $\hat{t} = t_s/(p-2)$, and the partition function therefore effectively depends on two parameters. The observation above suggests that $\hat{t}$ could have an interpretation as a true Kähler modulus. This follows from a different definition of the chiral gauge theory which is directly connected to the ordinary Yang-Mills one and leads to a closed topological string theory by itself [28].

The partition function $Z^{q\mathrm{YM},+}_{\hat{R}_1,\dots,\hat{R}_{|2g-2|}}(t_s)$ describes a truly open topological string theory with the worldsheets ending on the $|2g-2|$ stacks of D-branes that are geometrically represented by boundaries wrapping on the non-contractible cycles of the Calabi-Yau threefold. Nevertheless the boundaries are glued together, and the holomorphic and anti-holomorphic worldsheets match up on the D-branes. Worldsheets contributing to the total partition function are still closed, except that they are piecewise holomorphic or anti-holomorphic. A natural speculation is that boundary branes should be the analogue of the orientation-reversing tubes (and possibly even of the Ω-points) appearing in the Gross-Taylor string description of ordinary large N Yang-Mills theory.

The whole picture therefore seems very convincing and pointing towards a beautiful confirmation of the conjecture presented in [1]. It also suggests a natural embedding of two-dimensional Gross-Taylor string theory into a six-dimensional topological string theory, providing a new understanding of the QCD_2 string. However, there is a point that has been overlooked which could have interesting ramifications. In taking the large N limit it is possible to encounter phase transitions. The prototype of this kind of phenomenon was discovered long ago [46, 47] in the one-plaquette model of lattice gauge theory.

A well-known large N phase transition is the Douglas-Kazakov transition [24]. It concerns Yang-Mills theory on the sphere, a close relative of the relevant black hole ensembles discussed earlier. A strong-coupling phase, in which the theory admits the Gross-Taylor string description with its chiral-antichiral behaviour, is separated by a weak-coupling phase with gaussian field theoretical behavior. Two-dimensional Yang-Mills theory on S^2 is equivalent to a string theory only above a certain critical value of the effective t'Hooft coupling constant $\lambda = Ng^2A$ given by

$$\lambda_c = \pi^2 \ . \tag{2.29}$$

Instanton configurations induce the phase transition to the strong-coupling regime [25]. On the other hand, the entropy associated to certain classes of branched covering maps seems responsible for the divergence of the string perturbation series above the critical point [26, 27]. It is natural to expect that, if q-deformed large N Yang-Mills theory is related to the undeformed one, some of these features could find a place in the black holes/topological string scenario. Note that the partition function Z_{BH} describing the black-hole physics at large charges is not the topological string partition function, but rather its square modulus summed over a complicated set of boundary conditions. The final result is thus not an analytic function of the Kähler parameters, preventing in principle its analytical continuation below certain points. We will explore this possibility in the subsequent sections.

3. q-deformed Yang-Mills theory on S^2

In this section we will study the gauge theory structure of q-deformed Yang-Mills theory on S^2 at finite N. Even without reference to black hole physics, it is worthwhile studying this peculiar theory which connects instanton counting in four dimensions, topological strings in six dimensions and, as we explain momentarily, Chern-Simons theory in three dimensions.

3.1 General properties and the undeformed limit

Recall the definition of the partition function of q-deformed Yang-Mills theory on a genus g Riemann surface Σ_g given by

$$Z^q_{\mathrm{YM}} = \sum_R \dim_q(R)^{2-2g}\; q^{\frac{p}{2}\,C_2(R)}\;\; \mathrm{e}^{\,\mathrm{i}\,\theta\, C_1(R)}\,. \tag{3.1}$$

In deriving eq. (3.1) there are normalization ambiguities coming in part from a choice of regularization which corresponds [48] to additions of terms of the form

$$\alpha \int_{\Sigma_g} R + \beta \int_{\Sigma_g} K = \alpha\, \chi(\Sigma_g) + \beta' \tag{3.2}$$

to the action (2.14), where $\chi(\Sigma_g) = 2 - 2g$ is the Euler characteristic of Σ_g. The constants appearing here have been fixed in [6] by requiring consistency, in the large N limit, with topological string theory. They also have a different overall normalization in eq. (2.15). In their case the quantum dimension is multiplied by the quantity

$$S_{00} = \prod_{1\le i<j\le N} \left[q^{(j-i)/2} - q^{-(j-i)/2}\right]. \tag{3.3}$$

At finite N this term is only a function of g_s and N, and thus does not affect the main properties of the theory. Instead, it will assume an important role at large N in reproducing the correct contributions from constant maps to the topological string amplitudes.

We will momentarily ignore the regularization ambiguities and write the partition function with the factor S_{00}, so that

$$Z^q_{\mathrm{YM}} = \sum_R (S_{0R})^{2-2g}\; q^{\frac{p}{2}\,C_2(R)}\;\; \mathrm{e}^{\,\mathrm{i}\,\theta\, C_1(R)}\,. \tag{3.4}$$

In this form one recognizes the building blocks of U(N) Chern-Simons theory,

$$\dim_q(R) = \frac{S_{0R}}{S_{00}}\,, \tag{3.5}$$

where S_{RP} is the modular S-matrix of the U(N) WZW model on Σ_g at level k, with the important feature that the level k is not an integer. In fact, k is a purely imaginary number. At $p = 0 = \theta$ eq. (3.4) formally coincides with the partition function of Chern-Simons theory on the three-manifold $S^1 \times \Sigma_g$. The crucial difference is that we do not have here any truncation on the sum over the representations, which would usually be associated with the periodicity induced by integer-valued k. At more general values of p, it was claimed in [6] that eq. (3.4) should correspond to Chern-Simons theory on a circle bundle over Σ_g. A strong suggestion along these lines comes from observing that this Seifert manifold is the boundary the non-compact four-cycle C_4 wrapped by the D4-branes. The partition function of q-deformed Yang-Mills theory should be viewed as providing a definition of Chern-Simons theory at non-integer values of k through the coupling $g_s = 2\pi/(k+N)$. We will substantiate this argument in genus 0 by showing that part of the periodicity survives in the sum over representations R, allowing for the explicit determination of the relevant Chern-Simons gauge theory.

The other theory connected to eq. (3.1) is of course ordinary Yang-Mills theory. As we saw in the previous section, as $g_s \to 0$ the quantum dimension goes smoothly into the ordinary one. In order to recover the undeformed partition function (2.16), we have also to send $p \to \infty$ with

$$g_s p = a = g^2 A \tag{3.6}$$

fixed. This simple observation has some far reaching consequences. Because in this particular limit ordinary Yang-Mills theory is reproduced, we can look at the q-deformed theory as a peculiar a/p expansion and hope that some interesting characteristic survives coming down from $p = \infty$. We also expect that, in the large N-limit, the Gross-Taylor string theory emerges from the closed topological string theory on X. From the geometrical point of view, the limit $p \to \infty$ should be understood as some kind of limit in the Calabi-Yau manifold. Transition functions on fibers become highly singular and the geometry is expected to collapse effectively into a two dimensional one, reproducing eventually the topological sigma-model proposed in [49]. From the three-dimensional perspective, this limit was shown in [29] to reduce the Chern-Simons partition function on the pertinent Seifert fibration over Σ_g to that of ordinary Yang-Mills theory on Σ_g. From a practical point of view, taking this double-scaling limit and obtaining the well-known QCD$_2$ results will be a useful consistency check in our computations.

In order to perform concrete calculations, it is better to express the partition function directly in terms of Young tableaux variables. The irreducible representations R of the U(N) gauge group are easily constructed from irreducible representations $\hat{R}$ of SU(N) by assigning the U(1) charge Q. Its values are determined accordingly to the isomorphism $\mathrm{U}(N) = \mathrm{SU}(N) \times \mathrm{U}(1)/\mathbb{Z}_N$ as

$$R = (\hat{R}, Q)\,, \quad Q = n + r\,N\,, \tag{3.7}$$

where n is the number of Young tableau boxes contained in $\hat{R}$ and r is any integer. The first and second Casimir invariants of R can be written as

$$\begin{aligned} C_1(R) &= Q\,, \\ C_2(R) &= C_2(\hat{R}) + \frac{Q^2}{N}\,, \end{aligned} \tag{3.8}$$

where the quadratic Casimir of the SU(N) representation $\hat{R}$ is given by

$$C_2(\hat{R}) = n\,N + \sum_{i=1}^{N} \hat{n}_i(\hat{n}_i + 1 - 2i) - \frac{n^2}{N} \tag{3.9}$$

with $\hat{n}_i$ the lengths of the rows of the Young tableaux corresponding to $\hat{R}$ obeying the constraints $\hat{n}_1 \geq \hat{n}_2 \geq \cdots \geq \hat{n}_N \geq 0$ and $\sum_{i=1}^{N} \hat{n}_i = N$. Although the above parameterization is well-suited for deriving the string representation at large N, to explore the modular properties of the theory we will find it useful to switch to the set of integers n_i defined by

$$n_i = \hat{n}_i + r - i\,. \tag{3.10}$$

The range of n_i is $+\infty > n_1 > n_2 > \cdots > n_N > -\infty$. In this parameterization, the partition function of the q-deformed gauge theory on S^2 assumes the simple form

$$Z^q_{\rm YM}(g_s,p) = \frac{1}{N!} \sum_{n_i \in \mathbb{Z}} {\rm e}^{\,-\frac{g_s p}{2}\,(n_1^2+\cdots+n_N^2) + {\rm i}\,\theta(n_1+\cdots+n_N)} \prod_{1\leq i<j\leq N} \sinh^2\left(\frac{g_s}{2}\,(n_i - n_j)\right)\,. \tag{3.11}$$

In eq. (3.11) we have used the total symmetry of the summand in n_i to remove their ordering, and the fact that the quantum dimension vanishes for coincident n_i to extend the sum over all of $\mathbb{Z}^N$. We have disregarded some overall factor that combines with the renormalization ambiguities. As we have remarked in the previous section, the comparison with $\mathcal{N} = 4$ gauge theory on C_4 is not natural when the partition function is written in series in ${\rm e}^{\,-g_s}$. A modular transformation of (3.11) is required and, as a result, some important properties will be unveiled.

3.2 Instanton expansion

The geometrical meaning of the q-deformed theory on S^2 and its relation with Chern-Simons theory on a particular Seifert manifold becomes more transparent when we consider the dual description in terms of instantons of the gauge theory, provided by a modular transformation of the series (3.11). This is accomplished by means of a Poisson resummation. It is also an efficient way to investigate the behaviour of the theory at weak coupling g_s, as we shall explain later. We begin by taking the Fourier-transform of the function

$$\mathcal{F}(x_1,\ldots,x_N) = {\rm e}^{\,-\frac{g_s p}{4}\sum_{i=1}^{N} x_i^2} \prod_{1\leq i<j\leq N} \sinh\left(\frac{g_s}{2}\,(x_i - x_j)\right) \tag{3.12}$$

using the technique of orthogonal polynomials [50 − 52, 58]. Shifting the variables of integration as $x_i \to y_i + \frac{(1-N)}{p} + \frac{4\pi\,{\rm i}\,s_i}{g_s p}$, we have

$$\mathcal{S}(s_1,\ldots,s_N) = \int_{-\infty}^{\infty} {\rm d}x_1 \cdots {\rm d}x_N \;\; {\rm e}^{\,2\pi\,{\rm i}\sum_{i=1}^{N} s_i x_i}\, \mathcal{F}(x_1,\ldots,x_N)$$

$$= C_N \ \mathrm{e}^{\sum_{i=1}^N \frac{(g_s\,(N-1)-4\pi\,\mathrm{i}\,s_i)^2}{4\,g_s\,p}} \int_{-\infty}^{\infty} \mathrm{d}y_1 \cdots \mathrm{d}y_N \ \mathrm{e}^{-\frac{g_s p}{4} \sum_{i=1}^N y_i^2}$$
$$\times \prod_{1\le i<j\le N} \left(\mathrm{e}^{g_s y_i + \frac{4\pi\,\mathrm{i}\,s_i}{p}} - \mathrm{e}^{g_s y_j + \frac{4\pi\,\mathrm{i}\,s_j}{p}} \right) \tag{3.13}$$

where $C_N = 2^{-\frac{N(N-1)}{2}} \ \mathrm{e}^{-g_s \frac{N(1-N)^2}{2p}}$. We set $\mathrm{e}^{g_s y_i} = \mathrm{e}^{-2g_s/p} z_i$ to reduce the integral to the standard form

$$\mathcal{S}(s_1,\dots,s_N) = D_N \ \mathrm{e}^{\sum_{i=1}^N \frac{(g_s\,(N-1)-4\pi\,\mathrm{i}\,s_i)^2}{4\,g_s\,p}} \int_0^{\infty} \mathrm{d}z_1 \cdots \mathrm{d}z_N \ \mathrm{e}^{-\frac{p}{4g_s} \sum_{i=1}^N \log(z_i)^2}$$
$$\times \prod_{1\le i<j\le N} \left(z_i \ \mathrm{e}^{\frac{4\pi\,\mathrm{i}\,s_i}{p}} - z_j \ \mathrm{e}^{\frac{4\pi\,\mathrm{i}\,s_j}{p}} \right) \tag{3.14}$$

where $D_N = C_N g_s^{-N} \ \mathrm{e}^{-g_s N^2/p}$.

The factor $\mathrm{e}^{-\frac{p}{4g_s} \log(z)^2}$ is the natural measure for the Stieltjes-Wigert polynomials $W_k(z;q)$ [53] with the parameter q chosen to be $q = \mathrm{e}^{2g_s/p}$. Some useful details and properties of these polynomials are given in appendix A. We can expand the Vandermonde determinant in terms of these polynomials as

$$\prod_{1\le i<j\le N} \left(z_i \ \mathrm{e}^{\frac{4\pi\,\mathrm{i}\,s_i}{p}} - z_j \ \mathrm{e}^{\frac{4\pi\,\mathrm{i}\,s_j}{p}} \right)$$
$$= \sum_{\pi\in S_N} (-1)^{|\pi|} \, W_0(z_{\pi(1)} \ \mathrm{e}^{4\pi\,\mathrm{i}\,s_{\pi(1)}/p}, q) \cdots W_{N-1}(z_{\pi(N)} \ \mathrm{e}^{4\pi\,\mathrm{i}\,s_{\pi(N)}/p}, q) \,. \tag{3.15}$$

Substituting the representation (3.15) into eq. (3.14), we can perform exactly the integral in a closed form (see appendix A.2) to get

$$\mathcal{S}(s_1,\dots,s_N) = F_N \ \mathrm{e}^{\sum_{i=1}^N \frac{(g_s\,(N-1)-4\pi\,\mathrm{i}\,s_i)^2}{4\,g_s\,p}} \tag{3.16}$$
$$\times \sum_{\pi\in S_N} (-1)^{|\pi|} \, \hat{S}_0(-q^{(1-1)/2} \ \mathrm{e}^{4\pi\,\mathrm{i}\,s_{\pi(1)}/p}) \cdots \hat{S}_{N-1}(-q^{(1-N)/2} \ \mathrm{e}^{4\pi\,\mathrm{i}\,s_{\pi(N)}/p}) \,,$$

where $\hat{S}_k(z)$ are proportional to the Szegò polynomials (see appendix A) and

$$F_N = \left(\frac{4\pi}{g_s p}\right)^{\frac{N}{2}} \ \mathrm{e}^{-\frac{g_s(N-2)\,(N-1)\,N}{6\,p}} \ 2^{-\frac{(N-1)\,N}{2}} \,. \tag{3.17}$$

Recalling the definition of the Vandermonde determinant, eq. (3.16) can be rewritten as

$$\mathcal{S}(s_1,\dots,s_N) = F_N \ \mathrm{e}^{\sum_{i=1}^N \frac{(g_s\,(N-1)-4\pi\,\mathrm{i}\,s_i)^2}{4\,g_s\,p}} \prod_{1\le i<j\le N} \left(\mathrm{e}^{\frac{4\pi\,\mathrm{i}\,s_i}{p}} - \mathrm{e}^{\frac{4\pi\,\mathrm{i}\,s_j}{p}} \right) . \tag{3.18}$$

The final step in the Poisson resummation employs a well-known property of the convolution product under Fourier transformation to get

$$Z_{\rm YM}^{q,\rm inst}(s_1,\dots,s_N) = \int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \mathrm{d}x_N \ \mathrm{e}^{2\pi\,\mathrm{i} \sum_{i=1}^N s_i x_i} \, \mathcal{F}(x_1,\dots,x_N)^2$$

$$= \mathrm{e}^{-\frac{2\pi^2}{g_s p}\sum_{i=1}^N s_i^2}\ w_q^{\rm inst}(s_1,\dots,s_N) \tag{3.19}$$

where

$$\begin{aligned} w_q^{\rm inst}(s_1,\dots,s_N) = {} & \frac{1}{2}\left(\frac{2\pi}{g_s p}\right)^N\ \mathrm{e}^{-\frac{g_s\left(N^3-N\right)}{6p}} \int_{-\infty}^{\infty} \mathrm{d}z_1\cdots\mathrm{d}z_N\ \mathrm{e}^{-\frac{2\pi^2}{g_s p}\sum_{i=1}^N z_i^2} \\ & \times \prod_{1\le i<j\le N}\left[\cos\left(\frac{2\pi\ (s_i-s_j)}{p}\right)-\cos\left(\frac{2\pi\ (z_i-z_j)}{p}\right)\right] . \end{aligned} \tag{3.20}$$

The total partition function thereby turns out to be

$$Z_{\rm YM}^q(g_s,p) = \frac{1}{N!}\sum_{s_i\in\mathbb{Z}} \mathrm{e}^{-\frac{2\pi^2}{g_s p}\sum_{i=1}^N (s_i-\theta)^2}\ w_q^{\rm inst}(s_1,\dots,s_N) . \tag{3.21}$$

Note that the coupling parameter θ simply appears in the exponential factor as a constant shift in the integers s_i, as $w_q^{\rm inst}(s_1,\dots,s_N)$ depends only on their differences. We shall call this expression the instanton expansion of q-deformed Yang-Mills theory on S^2. This terminology mimicks that of the undeformed theory and will be justified presently.

The partition function of ordinary two-dimensional Yang-Mills theory can be computed exactly via a nonabelian generalization of the Duistermaat-Heckman theorem [43]. It is given by a sum over contributions localized at the classical solutions of the theory. For finite N the $\mathrm{U}(N)$ path integral is given by a sum over unstable instantons where each instanton contribution is given by a finite, but non-trivial, perturbative expansion. By "instantons" we mean solutions of the classical Yang-Mills field equations

$$\mathrm{d}_A{}^*F = \mathrm{d}^*F + \mathrm{i}\,[A,{}^*F] = 0 \tag{3.22}$$

which are not gauge transformations of the trivial solution $A=0$. This equation implies that the scalar field $f={}^*F$ is covariantly constant and may therefore be regarded as a constant element of the Lie algebra of the $\mathrm{U}(N)$ gauge group. The background curvature breaks the $\mathrm{U}(N)$ gauge symmetry to the subgroup of $\mathrm{U}(N)$ which commutes with f. By gauge invariance we may assume that f is an $N\times N$ real diagonal matrix. Its collection of eigenvalues has multiplicities N_k with $\sum_k N_k = N$. In other words, the solutions of (3.22) are labelled by partitions of the rank N of the gauge theory, and for a given partition $\{N_k\}$ the symmetry breaking is $\mathrm{U}(N)\to\prod_k\mathrm{U}(N_k)$. If $E\to\Sigma_g$ is the corresponding principal $\mathrm{U}(N)$-bundle of the gauge theory on Σ_g, then near each critical point it admits a decomposition $E=\bigoplus_k E_k$ into $\mathrm{U}(N_k)$ sub-bundles $E_k\to\Sigma_g$.

On the sphere S^2, the most general solution (up to gauge transformations) is given by

$$\big(A(z)\big)_{ij} = \delta_{ij}\ A^{(m_i)}(z) \tag{3.23}$$

where $A^{(m_i)}(z)$ is the Dirac monopole potential of magnetic charge m_i. The bundle splitting is described by taking $E_k = (\mathcal{L}^{\otimes m_k})^{\oplus N_k}$, where $\mathcal{L}\to S^2$ is the monopole line bundle (or equivalently the canonical line bundle over $\mathbb{P}^1$) which is classified by the Hopf fibration $S^3\to S^2$. The Yang-Mills action evaluated on such an instanton is given by

$$S_{\rm inst} = \frac{2\pi^2}{g^2 A}\sum_{i=1}^N m_i^2 . \tag{3.24}$$

Poisson resummation exactly provides the representation of ordinary Yang-Mills theory on S^2 in terms of instantons [54, 25]. Looking closer at eq. (3.21) we recognize a similar structure emerging. We observe the expected exponential of the "classical action" (at $\theta = 0$) $\mathrm{e}^{-\frac{2\pi^2}{g_s p}\sum_{i=1}^N s_i^2}$ and the fluctuations $w_q^{\rm inst}(s_1, \cdots, s_N)$ which smoothly reduce to the undeformed ones in the double scaling limit. The instanton representation is also useful to control the asymptotic behaviour of the partition function as $g_s \to 0$. In this limit, only the zero-instanton sector survives, the others being exponentially suppressed (for fixed p). In ordinary two-dimensional Yang-Mills theory the limiting partition function computes the volume of the moduli space of flat connections on the underlying Riemann surface and also intersection pairings on this moduli space [43]. For the case of S^2 the moduli space consists of a single point and the limit is trivial. Instead, in the q-deformed case we expect non-trivial geometrical structures emerge [29].

The arguments based on localization follow from the Yang-Mills action (2.14) in undeformed case wherein the scalar field Φ is non-compact. Varying the gauge field A in this action gives the equations of motion $\mathrm{d}_A\phi = 0$, implying as above that the classical solutions for Φ may be taken to be constant, diagonal real $N \times N$ matrices. Varying Φ gives the equation (at $\theta = 0$) $\Phi = f$, which thus implies the Yang-Mills equation (3.22) for the gauge field. However, this argument breaks down in the case that Φ is a compact $\mathrm{U}(N)$ scalar field. Heuristically, one should add to the non-compact scalar field f a sum over all image charges which render the classical solution $\Phi = f$ effectively compact. This means that, in addition to the sum over partitions of N which specify the given bundle splitting, there is an additional integer sum in each bundle component E_k. However, even this heuristic argument is sloppy, because the effect of the deformation is encoded in the measure for the scalar field.

We can gain some insight into the structure of the classical solutions of q-deformed Yang-Mills theory via the following equivalent reformulation in terms of generalized Yang-Mills theory [55]. The aim in this reformulation is to provide some insight on the classical solutions of the theory and to clarify the use of the term instanton. Let us start from the partition function expressed in the form (3.11). The quantum dimension of the representations can be exponentiated giving

$$\begin{aligned} Z^q_{\rm YM}(g_s,p) = \frac{1}{N!}\left(\frac{g_s}{2}\right)^2 \sum_{n_i\in\mathbb{Z}} \mathrm{e}^{-\frac{g_s p}{2}(n_1^2+\cdots+n_N^2)+\mathrm{i}\,\theta(n_1+\cdots+n_N)} \prod_{1\le i<j\le N}(n_i-n_j)^2 \\ \times \exp\left[-\sum_{i,j=1}^N \log\left(\frac{g_s(n_i-n_j)}{2\sinh\left(\frac{g_s}{2}(n_i-n_j)\right)}\right)\right] . \end{aligned} \tag{3.25}$$

The function that appears in the last exponential factor can rewritten using the expansion

$$\log\left(\frac{x}{\sinh(x)}\right) = \sum_{k=1}^\infty(-1)^k\,\frac{\zeta(2k)}{k}\,\left(\frac{x}{\pi}\right)^{2k} \tag{3.26}$$

to get

$$\sum_{i,j=1}^N \log\left[\frac{g_s(n_i-n_j)}{2\sinh(\frac{g_s}{2}(n_i-n_j))}\right]$$

$$= \sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2\pi)^{2k}\,k}\, g_s^{2k} \sum_{\ell=0}^{2k} (-1)^{k+\ell} \binom{2k}{\ell} \left(\sum_{i=1}^{N} n_i^{\ell}\right) \left(\sum_{j=1}^{N} n_j^{2k-\ell}\right) . \tag{3.27}$$

If we now introduce the diagonal matrix $\phi = g_s \ \mathrm{diag}\,(n_1, \dots, n_N)$, then we can finally rewrite (3.11) as

$$Z^q_{\mathrm{YM}}(g_s, p) = \frac{1}{N!} \left(\frac{g_s}{2}\right)^2 \sum_{n_i \in \mathbb{Z}} (n_i - n_j)^2 \ \ \mathrm{e}^{\,V(\phi) - \frac{g_s p}{2}\,\mathrm{Tr}(\phi^2) + \mathrm{i}\,\theta\,\mathrm{Tr}(\phi)} . \tag{3.28}$$

In this representation the only difference from the usual partition function of the undeformed theory is in the apparence of an auxilliary potential for the matrix ϕ given by

$$V(\phi) = -\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2\pi)^{2k}\,k} \sum_{\ell=0}^{2k} (-1)^{\ell+k} \binom{2k}{\ell} \mathrm{Tr}(\phi^{\ell})\,\mathrm{Tr}(\phi^{2k-\ell}) . \tag{3.29}$$

These formal manipulations suggest that the partition function of the q-deformed theory can be derived from a generalized (but undeformed) Yang-Mills theory with action

$$\begin{aligned} S^{\mathrm{gen}}_{\mathrm{YM}_2} = {} & \frac{1}{g_s} \int_{S^2} \mathrm{Tr}(\Phi\,F) - \frac{p}{2g_s} \int_{S^2} \mathrm{Tr}(\Phi^2)\ K + \frac{\theta}{g_s} \int_{S^2} \mathrm{Tr}(\Phi)\ K \\ & + \frac{1}{8\pi} \int_{S^2} V(\Phi)\ R\,, \end{aligned} \tag{3.30}$$

where R is the Ricci curvature two-form of a fixed metric on S^2. This representation holds in fact for all genera. It assumes that the path integral for the action (3.30) can be localized onto a sum over $\mathrm{U}(1)^N$ bundles for the gauge field, while the integration over Φ is localized onto constant field configurations ϕ. The main difference between this representation and the formulation of [6] is that the scalar field Φ is not a periodic variable and its measure in the path integral is the standard one. Moreover, the non-abelian localization of the path integral is not altered by perturbing the ordinary Yang-Mills action by deformations $V(\Phi)$ which depend only on the scalar field. Thus the partition function will localize again onto critical points, this time of the action functional (3.30). By varying the action with respect to the gauge field, one finds again that the field Φ is covariantly constant. However, now the equation of motion for Φ itself involves a complicated non-linear relation between the scalar field and the gauge field. Using gauge invariance to diagonalize Φ, by applying the covariant derivative to the equation of motion for Φ we conclude also that $f = {}^*F$ is covariantly constant and again obeys the Yang-Mills equation (3.22). The solutions for the gauge potential are therefore again of the form (3.23). Thus the critical points of the q-deformed theory (or equivalently of our generalized QCD_2) behave exactly as the instantons of ordinary two-dimensional Yang-Mills theory.

The effect of the deformation is now completely encoded in the auxiliary potential for the field Φ. However, its precise dynamical role is not completely clear. To understand this problem, let us write down explicitly the equations of motion for the scalar field when the gauge group is U(2). Diagonalizing the fields as described above to write them as

$$f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} , \qquad \Phi = \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} , \tag{3.31}$$

the two equations of motion for Φ have the form

$$\begin{aligned}
&\frac{f_1}{g_s}-\frac{p}{g_s}\,\phi_1-\frac{1}{\phi_1-\phi_2}+\frac{1}{2}\cot\left(\frac{\phi_1-\phi_2}{2}\right)=0\,,\\
&\frac{f_2}{g_s}-\frac{p}{g_s}\,\phi_2+\frac{1}{\phi_1-\phi_2}-\frac{1}{2}\cot\left(\frac{\phi_1-\phi_2}{2}\right)=0\,,
\end{aligned} \tag{3.32}$$

where we have set the θ-angle to zero. Due to the branch structure of the cotangent function, these equations imply that for each monopole configuration of the gauge fields there exists infinitely many equivalent solutions for the scalar fields. This is likely to be related to the image charges mentioned above, and agrees with the form of the instanton expansion that we derive below.

We close this subsection by returning to the relation with the $\mathcal{N}=4$ topological gauge theory. Eq. (3.21) should be equivalent to the formula obtained in [6] for the modular transformation of $Z^q_{\rm YM}$. They performed a weak check for $p=1,2$, confronting the general structure of their expressions with some results appearing in the literature [40]. Unfortunately, there are very few results for the Euler characteristic of the moduli space of instantons on $C_4=\mathcal{O}(-p)\to\mathbb{P}^1$ for generic p. In confronting the two theories, eq. (3.21) should really reproduce the instanton actions and their fluctuation determinants which are represented here in terms of integrals. In the following, we shall try to understand better the fluctuations around the instanton $(s_1,\dots,s_N)$, i.e. the behaviour of the integral

$$\int_{-\infty}^{\infty}{\rm d}z_1\cdots{\rm d}z_N~{\rm e}^{-\frac{2\pi^2}{g_sp}\sum_{i=1}^N z_i^2}\prod_{1\le i<j\le N}\left[\cos\left(\frac{2\,\pi\,(s_i-s_j)}{p}\right)-\cos\left(\frac{2\,\pi\,(z_i-z_j)}{p}\right)\right]\,. \tag{3.33}$$

It can be localized onto a finite interval by using the periodicity of the product appearing in the integrand.

We now Poisson resum the generic series and write it in terms of an elliptic theta-function as

$$\sum_{n_i\in\mathbb{Z}}{\rm e}^{-\frac{2\pi^2p}{g_s}\,(z_i-n_i)^2}=\sqrt{\frac{g_s}{2\,\pi\,p}}\sum_{m_i\in\mathbb{Z}}{\rm e}^{-\frac{g_s}{2p}\,m_i{}^2+2\pi\,{\rm i}\,m_iz_i}=\sqrt{\frac{g_s}{2\,\pi\,p}}\,\vartheta_3\Big(\frac{{\rm i}\,g_s}{2\pi\,p}|z_i\Big)\,. \tag{3.34}$$

Then the fluctuation integral is given by

$$\left(\frac{g_sp}{2\pi}\right)^{\frac{N}{2}}\int_0^1{\rm d}z_1\cdots{\rm d}z_N\prod_{i=1}^N\vartheta_3(\frac{{\rm i}\,g_s}{2\pi p}|z_i)\prod_{1\le i<j\le N}\left[\cos\left(\frac{2\pi(s_i-s_j)}{p}\right)-\cos\big(2\pi\,(z_i-z_j)\big)\right]\,. \tag{3.35}$$

The product can be equivalently written in terms of Vandermonde determinants $\Delta(y_i)$ as

$$\begin{aligned}
&\prod_{1\le i<j\le N}\left[\cos\left(\frac{2\,\pi\,(s_i-s_j)}{p}\right)-\cos\big(2\,\pi~(z_i-z_j)\big)\right]\\
&\qquad=\frac{{\rm e}^{-\frac{2\pi\,{\rm i}}{p}\,(N-1)\sum_{l=1}^N s_l}}{2^N}\,\Delta\left({\rm e}^{\,2\pi\,{\rm i}\,(\frac{s_i}{p}-z_i)}\right)\,\Delta\left(e^{2\pi\,{\rm i}\,(\frac{s_i}{p}+z_i)}\right)\,.
\end{aligned} \tag{3.36}$$

Thus the generic fluctuation term appears in the form

$$\left(\frac{g_s p}{2\pi}\right)^{\frac{N}{2}} \frac{\mathrm{e}^{-\frac{2\pi\,\mathrm{i}}{p}\,(N-1)\sum_{l=1}^{N} s_l}}{2^N} \int_0^1 \mathrm{d}z_1 \cdots \mathrm{d}z_N \, \prod_{i=1}^{N} \vartheta_3\big(\frac{\mathrm{i}\, g_s}{2\pi\, p}|z_i\big)$$
$$\times \, \Delta\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,(\frac{s_i}{p}-z_i)}\right) \Delta\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,(\frac{s_i}{p}+z_i)}\right)\,, \tag{3.37}$$

with the Fourier transforms substituted by compact integrals and the elliptic theta-function playing the role of the measure. In this form, it strongly resembles the integrals appearing in [57, 56] where the B-model partition function of open topological strings on the resolved conifold was expressed in terms of a unitary matrix model.

3.3 Relation to Chern-Simons theory on Lens spaces

There is a very natural connection between the proposal of [6] and Chern-Simons theory on a Lens space. As explained before, the instanton counting in the $\mathcal{N}=4$ topologically twisted gauge theory living on the four-manifold $C_4 = \mathcal{O}(-p) \to \mathbb{P}^1$ localizes onto the two-dimensional deformed gauge theory on the base of this holomorphic line bundle. The total space C_4 of the fibration is an ALE space which asymptotes to the orbifold $\mathbb{C}^2/\mathbb{Z}_p$, where $\mathbb{Z}_p$ acts on $(z,w)\in\mathbb{C}^2$ by $(z,w)\mapsto \mathrm{e}^{\,2\pi\,\mathrm{i}\,k/p}\,(z,w)$ for $k=0,1,\dots,p-1$ [62]. More precisely, as a scalar-flat Kähler four-manifold $C_4 \cong D(\mathcal{L}^{\otimes -p})$ is the disk bundle of the dual of the monopole line bundle over S^2 of magnetic charge $-p$. The boundary of this manifold is a circle bundle over the sphere, $\partial C_4 \cong S(\mathcal{L}^{\otimes -p})$, which is diffeomorphic to the Lens space $L_p = S^3/\mathbb{Z}_p \cong S(\mathcal{L}^{\otimes -p})$, where we regard $S^3 \subset \mathbb{C}^2$ with the embedding $|z|^2+|w|^2=1$. Another interesting perspective has been discussed in [60] where the partition function of ordinary Yang-Mills on a cylinder, with trivial boundary conditions at the two ends of the cylinder, was shown to be equivalent to the Chern-Simons partition function on $S^3/\mathbb{Z}_p$. In [61] contact was made with the q-deformed theory on the sphere.

The relation to Chern-Simons gauge theory on $\partial C_4 \simeq L_p$ now follows classically from the action (2.7). An instanton excitation on a generic four manifold with boundary is related to a Chern-Simons theory on the boundary through

$$\int_{C_4} \mathrm{Tr}\, F\wedge F = \int_{C_4} \mathrm{Tr}\ \mathrm{d}\left(A\wedge \mathrm{d}A + \frac{2}{3}\,A\wedge A\wedge A\right) = \int_{\partial C_4} \mathrm{Tr}\left(A\wedge\mathrm{d}A + \frac{2}{3}\,A\wedge A\wedge A\right)\,. \tag{3.38}$$

Of course, this term only describes the pure topological sector of the q-deformed gauge theory described by the BF lagrangian $\mathrm{Tr}(\Phi\, F)$, which arises in the weak-coupling limit. The remaining terms in the four-dimensional action, along with the massive deformation $\mathrm{Tr}\ \Phi^2$, conspire to give a more complicated three-dimensional gauge theory than just Chern-Simons theory. According to [29], the three dimensional Chern-Simons theory defined on a circle fibration over a Riemann surface is naturally related to a two dimensional gauge theory on the base. We will describe this relationship more explicitly later on in this section. For now, we will show how Chern-Simons theory on the Lens space L_p emerges from the two dimensional point of view. We will set $\theta=0$ as the θ-angle is irrelevant for the present discussion.

Let us focus our attention on the partition function of the q-deformed theory expressed in terms of instantons. A remarkable property of the fluctuations is that they do not depend really on the integers s_i but only their values modulo p. In other words, two instanton configurations $\vec{s} = (s_1, \ldots, s_N)$ and $\vec{s'} = (s'_1, \ldots, s'_N)$ that differ by $p \cdot \vec{h}$ possess the same fluctuation factor, where $\vec{h}$ is any vector with N integer entries. The number of independent fluctuations is then always finite and less than p^N. It is natural to organize the partition function by factorizing the independent fluctuations. This can be easily achieved by writing each integer s_i as

$$s_i = p\,\ell_i + \hat{s}_i \tag{3.39}$$

where $l_i \in \mathbb{Z}$ $\hat{s}_i \in \{0, 1, \ldots, p-1\}$. Then the instanton expansion of the partition function $Z^q_{\rm YM}$ can be written as

$$Z^q_{\rm YM} = \frac{1}{N!} \sum_{\hat{s}_i=0}^{p-1} \sum_{\ell_i=-\infty}^{\infty} \; \mathrm{e}^{-\frac{2\pi^2}{g_s p} \sum_{i=1}^N (p\,\ell_i + \hat{s}_i)^2} \; w_q^{\rm inst}(\hat{s}_1, \ldots, \hat{s}_n)\,. \tag{3.40}$$

Two fluctuations $w_q^{\rm inst}(\hat{s}_1, \ldots, \hat{s}_n)$ that differ only by a reordering of the $\hat{s}_i$ give again the same contribution. From this observation we conclude that independent fluctuations are completely characterized by the set of non-negative integers N_k which count the number of times the integers $k \in \{0, \ldots, p-1\}$ appears in the string $(\hat{s}_1, \ldots, \hat{s}_n)$. We have the obvious sum rule $\sum_k N_k = N$.

The simplest way to eliminate the huge degeneracy in the fluctuation factors is to reorder the integers $\hat{s}_i$ and sum only over those configurations with $\hat{s}_1 \le \hat{s}_2 \le \cdots \le \hat{s}_N$. In this way we arrive at an elegant sum over ordered partitions $\{N_k\}$ of N into p parts, and we obtain

$$\begin{aligned} Z^q_{\rm YM} \;=\; & \sum_{\substack{N_1,\ldots,N_k \\ \sum_k N_k = N}} \frac{1}{\prod_k N_k!} \sum_{\ell_i \in \mathbb{Z}} \exp\left[-\frac{2\pi^2}{g_s p} \left(\sum_{i=1}^{N_0} (p\,\ell_i)^2 + \sum_{i=N_0+1}^{N_0+N_1} (p\,\ell_i + 1)^2 + \cdots \right.\right. \\ & \left.\left. + \sum_{i=N_0+\cdots+N_{p-2}+1}^{N} (p\,\ell_i + p - 1)^2 \right)\right] \; w_q^{\rm inst}(\underbrace{0, \ldots, 0}_{N_0}, \ldots, \underbrace{p-1, \ldots, p-1}_{N_{p-1}}) \\ = \; & \sum_{\substack{N_1,\ldots,N_k \\ \sum_k N_k = N}} \frac{1}{\prod_k N_k!} \prod_{k=0}^{p-1} \vartheta_3 \left(\frac{2\pi i p}{g_s} \middle| \frac{2\pi i k}{g_s} \right)^{N_k} \\ & \times \exp\left(-\frac{2\pi^2}{g_s p} \sum_{m=0}^{p-1} N_m m^2 \right) \; w_q^{\rm inst}(\underbrace{0, \ldots, 0}_{N_0}, \ldots, \underbrace{p-1, \ldots, p-1}_{N_{p-1}})\,. \end{aligned} \tag{3.41}$$

This is the central result of this section. We recognize in the second line the partition function of $\mathrm{U}(N)$ Chern-Simons gauge theory on the Lens space L_p in a non-trivial vacuum given by [58, 59]

$$Z^p_{\rm CS}\big(\{N_k\}\big) = \exp\left(-\frac{2\pi^2}{g_s p} \sum_{m=0}^{p-1} N_m m^2 \right) \; w_q^{\rm inst}(\underbrace{0, \ldots, 0}_{N_0}, \ldots, \underbrace{p-1, \ldots, p-1}_{N_{p-1}})\,. \tag{3.42}$$

The critical points of the U(N) Chern-Simons action on the Seifert manifold L_p are flat connections which are classified by the embeddings of the first fundamental group into U(N). Since the $\mathbb{Z}_p$-action on S^3 used to define L_p is free, one has $\pi_1(L_p) = \mathbb{Z}_p$. The cyclic generator h of this fundamental group can be taken to be any loop which is the projection of a path on the universal cover $S^3 \to L_p$ connecting two points that are related by the $\mathbb{Z}_p$-action. The critical points are therefore given by discrete $\mathbb{Z}_p$-valued flat connections. They are easily described by choosing N-component vectors with entries taking values in $\mathbb{Z}_p$. Because the residual Weyl symmetry S_N of the U(N) gauge group permutes the different components, the independent choices are in correspondence with the partitions $\{N_k\}$. The possible vacua of the gauge theory are in one-to-one correspondence with the choices of flat connections. The full partition function of Chern-Simons theory involves summing over all the flat connections, and in fact the exact answer that can be obtained from the relation with the WZW model [63, 64] gives such a sum. Nevertheless, due to the fact that the flat connections here are isolated points, it is not difficult to extract the particular contribution of a given vacuum which coincides with eq. (3.42).

The relation we have found should be understood as an analytical continuation to imaginary values of the Chern-Simons coupling constant k by identifying

$$g_s p = \frac{2\pi\,\mathrm{i}}{k+N}\,, \tag{3.43}$$

in full agreement with the discussion of section 3.1. With this distinction in mind, we can write down the partition function of q-deformed Yang-Mills theory in the suggestive form

$$Z^q_{YM} = \sum_{\{N_k\}} \prod_{k=0}^{p-1} \frac{\theta_3\left(\frac{2\pi\,\mathrm{i}\,p}{g_s}\Big|\frac{2\pi\,\mathrm{i}\,k}{g_s}\right)^{N_k}}{N_k!}\; Z^p_{\mathrm{CS}}\big(\{N_p\}\big)\,. \tag{3.44}$$

The instanton contributions appear organized in definitive way. Some of them, and in particular the ones having the lowest classical action for each class in $\{N_k\}$, appear in the analytic continuation of $Z^p_{\mathrm{CS}}(\{N_k\})$. The instanton action, in this case, is exactly the Chern-Simons action evaluated on its critical point (the flat connection). The other instantons belonging to the class determined by $\{N_k\}$ are instead contained in the theta-function contributions multiplying the fundamental Chern-Simons partition function in the given background. It is tempting to speculate that they are related to the presence of the periodic measure in the path integral. A simple indication of this comes from the limit $g_s \to 0$. We expect that in this regime the effective periodicity goes to ∞ and the theory "decompactifies". In this limit $\vartheta_3 \to 1$ and the q-deformed partition function coincides with the total Chern-Simons partition function on L_p summed over all the non-trivial flat-connections.

The fact that Chern-Simons theory on Seifert manifolds localizes around flat connections in the same way that Yang-Mills theory localizes around instantons has been recently shown in [29]. The computation of quantum fluctuations around flat connections resembles in many respects the calculation around non-trivial instantons in Yang-Mills theory. This is in completely harmony with what we find here and it opens up the possibility to understand even the higher-instanton contributions as coming from classical solutions of q-deformed Yang-Mills theory. We will now describe more precisely how a flat connection on the three-manifold L_p is related to a generic instanton on S^2. Three-dimensional flat connections can be obtained from two dimensional instantons by assigning a fixed holonomy along the generator h of $\pi_1(L_p)$. The general correspondence on any Seifert fibration was proven in [65] for the global minima of the two-dimensional Yang-Mills action (the central connections corresponding to the trivial partition having only one component $N_1 = N$). In the case of the Lens space L_p the correspondence can be rephrased in a particularly simple fashion as follows.

Consider the monopole line bundle $\mathcal{L}$ over the sphere. We can form the bundle $\mathcal{L}^{\otimes -p}$ endowed with the monopole connection $A^{(-p)}$ of magnetic charge $-p$. We can describe a constant curvature instanton on S^2 with the two dimensional data (E, A), where E is a principal $\mathrm{U}(N)$-bundle over S^2 with first Chern number $q \in \{0, 1, \ldots, N-1\}$ and A is a connection on E of curvature $F_A = \mathbf{1}_N \, \frac{q}{N} \, F_{A^{(-p)}}$. The two dimensional connections are in one-to-one correspondence with flat connections on the circle bundle $S(\mathcal{L}^{\otimes -p}) = L_p$ with gauge group $\mathrm{SU}(N)$ and fixed holonomy $\mathrm{e}^{-2\pi \,\mathrm{i}\, q/N} \, \mathbf{1}_N$ in the center of $\mathrm{SU}(N)$ around the fiber. The three dimensional flat connection can be explicitly constructed in terms of the two dimensional constant curvature connection as follows. Let $\pi : S(\mathcal{L}^{\otimes -p}) \to S^2$ be the bundle projection. The pull-back under this projection naturally lifts to three dimensions the two dimensional data (E, A). Moreover, it can be used to define the trivial bundle $\pi^* \mathcal{L}^{\otimes -p} \cong S(\mathcal{L}^{\otimes -p}) \times \mathbb{C}$ with connection $a = \frac{1}{N} \, \pi^* A_0$, to which we have the freedom to add any trivial connection. With these data we can construct the bundle $E' = \pi^* E \otimes \pi^* \mathcal{L}^{\otimes -pq}$ over L_p whose curvature vanishes by construction The structure group of E' is $\mathrm{SU}(N)$, because the determinant line bundle $\det E' = \det \pi^* E \otimes (\pi^* \mathcal{L})^{\otimes -pqN}$ is endowed with a vanishing connection and the holonomy around the fiber is in the center of $\mathrm{SU}(N)$ since by construction $\pi^* A_0$ has trivial holonomy around the fiber of $L_p \to S^2$.

The above construction can be easily generalized to generic instantons corresponding to two dimensional connections which are not proportional to the identity $\mathbf{1}_N$. Let us suppose that the instanton is specified by a partition $\{N_k\}$ of N which corresponds to the gauge symmetry breaking $\mathrm{U}(N) \to \prod_k \mathrm{U}(N_k)$. In the vicinity of such an instanton the data (E, A) decompose as $E = \bigoplus_k E_k$ and $A = \bigoplus_k A^{(m_k)}$. The argument above can then be applied to each sub-bundle E_k with its central Yang-Mills connection $A^{(m_k)}$ to lift the most general reducible connection in two dimensions to a three dimensional Chern-Simons critical point. The main difference now is that the holonomy around the circle fiber is not in general an element of the center of $\mathrm{SU}(N)$ but will respect the symmetry breaking pattern. The holonomy is given by a block diagonal matrix whose entries are all of the form $e^{-2\pi \,\mathrm{i}\, q_k/N_k}$ where N_k is the rank of the corresponding subgroup and $0 \le q_k < N_k$.

A flat connection on a circle bundle over a Riemann surface with a holonomy around the fiber non-trivial but still proportional to the identity, that arises as the pull-back from a central $\mathrm{U}(N)$-connection on the base manifold, can be seen as the pull-back of a *flat* two dimensional connection on a non-trivial bundle whose structure group is $\mathrm{U}(N)/\mathrm{U}(1) \cong \mathrm{SU}(N)/\mathbb{Z}_N$. In other words, a constant curvature connection, which can be locally seen as arising from an element of the center of the structure group, becomes trivial when we divide the gauge group $\mathrm{U}(N)$ by its center $\mathrm{U}(1)$. However, a flat connection on a nontrivial $\mathrm{U}(N)/\mathrm{U}(1)$-bundle over the base Riemann surface can be described as a flat connection on a *trivial* $\mathrm{SU}(N)$-bundle with given monodromy in $\mathbb{Z}_N$ around an arbitrary point on the base. Thus the correspondence between two dimensional instantons and Chern-Simons critical points can be equivalently rephrased as a correspondence between three dimensional flat connections and two dimensional flat connections with non-trivial monodromy around an arbitrary point. This description explains the coincidence of the instanton expansion above in the weak-coupling limit, which effectively picks out the zero-instanton contributions, and the sum over vacua of the Chern-Simons partition function on L_p. The argument does not generalize to higher-instantons which instead give non-trivial contributions to the theta-functions appearing. The full effect of the q-deformation is captured by these functions.

We close this section by returning to the relation with topological string theory. The partition function of Chern-Simons theory on $L_p = S^3/\mathbb{Z}_p$ in the background

$$u\big(\{N_k\}\big) = (\underbrace{0,\dots,0}_{N_0},\dots,\underbrace{p-1,\dots,p-1}_{N_{p-1}}) \tag{3.45}$$

is the perturbative open topological string theory on the total space of the cotangent bundle T^*L_p of the Lens space, with $N_0, N_1, \dots, N_{p-1}$ topological D3-branes wrapped around the non-trivial cycles of L_p. The partition function $Z^p_{\mathrm{CS}}(\{N_k\})$ has been obtained in [59] from a matrix-model computation associated to the topological B-model on the mirror of L_p. The large N limit and the closed topological string theories emerging from the relevant geometric transition have been discussed in [59, 66, 67], assuming a fixed configuration of branes. The appearance of $Z^p_{\mathrm{CS}}(\{N_p\})$ in q-deformed Yang-Mills theory could suggest a topological open string underlying the theory and one could wonder if the geometry of X, in the large N limit, can be understood as a result of a geometric transition. There are a couple of basic differences which should be taken into account. Firstly, the N D3-branes are not fixed but the q-deformed theory sums over all possible wrappings. Secondly, the theta-function contributions do not appear in the perturbative topological string amplitudes.

4. Instanton driven large N phase transition

We are ready now to discuss the large N limit of q-deformed Yang-Mills theory on the sphere. We have expressed the partition function as a sum over classical solutions in a way that the limit $g_s \to 0$ is quite transparent and well related to the quantum field theoretical degrees of freedom. Let us see the fate of the instanton contributions at large N.

4.1 Instanton contributions at large N

We start by defining the relevant parameters to be held fixed as $N \to \infty$ by

$$t = g_s N\,, \quad a = g_s p N = p\,t\,. \tag{4.1}$$

Henceforth we will set $\theta = 0$. In terms of these variables, the partition function assumes the form

$$Z^q_{\mathrm{YM}} = \frac{1}{N!}\sum_{s_i\in\mathbb{Z}} \mathrm{e}^{-\frac{2\pi^2 N}{a}\sum_{i=1}^N s_i^2}\, w_q^{\mathrm{inst}}(s_i; N, a, t)\,. \tag{4.2}$$

We immediately recognize that all the non-trivial instanton contributions are nonperturbative in the $\frac{1}{N}$ expansion, being naively exponentially suppressed, suggesting that the theory could reduce in this limit to the zero-instanton sector. In order for this possibility to be correct, one should control the fluctuation factors. Let us recall what happens in ordinary Yang-Mills theory. In that case the corrections due to the contribution of instantons to the free energy were calculated in [25]. They find that while in the weak-coupling phase this contribution is exponentially small, it blows up as the phase transition point is approached. The transition occurs when the entropy of instantons starts dominating over their Boltzmann weight $\mathrm{e}^{-S_{\rm inst}}$. The order of perturbation theory required to compute the fluctuations around the instantons grows with N, and as $N \to \infty$ one gets contributions from all orders which compete with the exponential suppression. The density of instantons then goes from 0 at weak coupling to ∞ at strong coupling.

In principle, the q-deformed theory on S^2 could experience the same fate. In eq. (4.2) the Boltzmann weights are the same as in the undeformed case, and only the structure of the fluctuations is changed by the deformation. As written in eq. (3.37), the instanton factors exhibit an "entropic" behavior because in the large N limit one has $\theta_3 \to 1$, and the divergence appears thanks to the huge number of (compact) integrals to be performed.

One way to detect the presence of a phase transition at $N \to \infty$ is to look for a region in the parameter space where the one-instanton contribution dominates the zero-instanton sector [25]. In our case the ratio of the two contributions is given by

$$\begin{aligned} \mathrm{e}^{F_0} &= \frac{\mathcal{Z}^{1-\rm inst}}{\mathcal{Z}^{0-\rm inst}} \\ &= \frac{\int \prod_{i=1}^{N} \mathrm{d}z_i \ \mathrm{e}^{-\frac{8\pi^2 N}{a}\sum_{i=1}^N z_i^2} \prod_{j=2}^{N} \left[\sin^2\!\left(\frac{2\pi t}{a}(z_1-z_j)\right)-\sin^2\!\left(\frac{\pi t}{a}\right)\right] \prod_{\substack{i<j\\ i\geq 2}} \sin^2\!\left(\frac{2\pi t}{a}(z_i-z_j)\right)}{\mathrm{e}^{\frac{2\pi^2 N}{a}} \int \prod_{i=1}^{N} \mathrm{d}z_i \ \mathrm{e}^{-\frac{8\pi^2 N}{a}\sum_{i=1}^N z_i^2} \prod_{1\leq i<j\leq N} \sin^2\!\left(\frac{2\pi t}{a}(z_i-z_j)\right)}, \end{aligned} \tag{4.3}$$

where we have dropped an irrelevant normalization factor. We shall employ the saddle-point technique to compute the above ratio. Consider the integral

$$\mathcal{Z}^{0-\rm inst} = 2\int \prod_{i=1}^{N} \mathrm{d}z_i \ \mathrm{e}^{-\frac{8\pi^2 N}{a}\sum_{i=1}^N z_i^2} \prod_{1\leq i<j\leq N} \sin^2\left(\frac{2\pi\, t\,(z_i-z_j)}{a}\right). \tag{4.4}$$

In the large N limit, it is dominated by the solutions of the saddle-point equation

$$z_i = \frac{t}{4\pi\, N}\sum_{j\neq i} \cot\left(\frac{2\pi\, t\,(z_i-z_j)}{a}\right) \tag{4.5}$$

Introduce the density

$$\rho(z) = \frac{1}{N}\sum_{i=1}^{N} \delta(z-z_i) \tag{4.6}$$

to rewrite eq. (4.5) as

$$z = \frac{t}{4\pi}\int \mathrm{d}w\ \rho(w)\ \cot\left(\frac{2\pi\, t\,(z-w)}{a}\right). \tag{4.7}$$

A convenient way to analyse this integral equation is to notice that under the change of variables $t \mapsto \mathrm{i}\,\hat{t}$ it maps to

$$\frac{8\pi^2}{a}\,z = \frac{2\pi\,\hat{t}}{a}\int \mathrm{d}w\;\rho(w)\;\coth\left(\frac{2\pi\,\hat{t}\,(z-w)}{a}\right). \tag{4.8}$$

Fortunately, we do not have to solve eq. (4.8) directly, because we can map this equation to the one already solved in [68] in the context of topological strings and matrix models. Borrowing the solution we get

$$\rho(z) = \frac{4}{\hat{t}}\;\arctan\left(\sqrt{\frac{\mathrm{e}^{\,\hat{t}^2/a}}{\cosh^2(\frac{2\pi\,\hat{t}\,z}{a})} - 1}\right). \tag{4.9}$$

Now we analytically continue $\hat{t} \to -\,\mathrm{i}\,t$, giving the final result

$$\rho(z) = \frac{4}{t}\,\mathrm{arctanh}\left(\sqrt{1 - \frac{\mathrm{e}^{\,-t^2/a}}{\cos^2(\frac{2\pi\,t\,z}{a})}}\right) = \frac{4}{t}\,\mathrm{arccosh}\left(\mathrm{e}^{\,t^2/2a}\;\cos\left(\frac{2\pi t z}{a}\right)\right). \tag{4.10}$$

The support of the density $\rho(z)$ is easily determined by imposing the condition $\cos^2(\frac{2\pi\,t\,z}{a}) - \mathrm{e}^{\,-t^2/a} \geq 0$, giving

$$|z| < \frac{a}{2\pi\,t}\;\arccos(\,\mathrm{e}^{\,-t^2/2a}). \tag{4.11}$$

In the large N limit, the ratio $\mathrm{e}^{\,F_0}$ is completely determined by this distribution as

$$\mathrm{e}^{\,F_0} = \int \mathrm{d}z\;\exp\left(-\frac{8\pi^2 N z^2}{a} - \frac{2\pi^2 N}{a} + N\int \mathrm{d}w\;\log\left[\frac{\sin^2(\frac{\pi t}{a}) - \sin^2\left(\frac{2\pi t(z-w)}{a}\right)}{\sin^2\left(\frac{2\pi t(z-w)}{a}\right)}\right]\rho(w)\right). \tag{4.12}$$

We can safely assume that the integral over z is peaked around $z = 0$ in the large N limit. Then eq. (4.12) reduces to

$$\mathrm{e}^{\,F_0} = \exp\left(-\frac{2\pi^2 N}{a} + N\int \mathrm{d}w\;\;\log\left[\frac{\sin\left(\frac{2\pi\,t(1/2+w)}{a}\right)\,\sin\left(\frac{2\pi\,t(1/2-w)}{a}\right)}{\sin^2\left(\frac{2\pi t\,w}{a}\right)}\right]\rho(w)\right). \tag{4.13}$$

This integral can be computed in closed form but we prefer to follow a different route. The basic quantity to consider is

$$G(x) = \int \mathrm{d}w\;\rho(w)\;\log\left|\sin\left(\frac{2\pi\,t(x-w)}{a}\right)\right| - \frac{4\pi^2}{a}\,x^2. \tag{4.14}$$

The expression (4.13) can then be written in terms of $G(x)$ as

$$\mathrm{e}^{\,F_0} = \exp\left[N\left(G\left(\frac{1}{2}\right) + G\left(-\frac{1}{2}\right) - 2G(0)\right)\right] = \exp\left[2N\left(G\left(\frac{1}{2}\right) - G(0)\right)\right], \tag{4.15}$$

where we have used the reflection symmetry $G(x) = G(-x)$.

The function (4.14) has an interesting behaviour. Both $G(x)$ and $G'(x)$ are continuous even when x reaches one of the endpoints of the support of the density ρ which vanishes as $\sqrt{x}$ there. The integral (4.14) is thus convergent also at $x = \pm \frac{a}{2\pi\, t} \arccos(\mathrm{e}^{-t^2/2a})$. Moreover, $G(x)$ is constant when x lies in the support region (4.11) of the density. By taking the derivative with respect to x we obtain

$$G'(x) = \frac{2\pi\, t}{a} \int \mathrm{d}w\; \rho(w)\; \cot\left(\frac{2\pi\, t(x-w)}{a}\right) - \frac{8\pi^2}{a}\, x = 0 \tag{4.16}$$

due to the saddle-point equation. For x outside of the support region (4.11), the saddle-point equation no longer holds and we cannot conclude that $G(x)$ is constant. In order to understand what happens in this instance it is sufficient compute the second derivative

$$G''(x) = -\left(\frac{2\pi\, t}{a}\right)^2 \int \mathrm{d}w\; \frac{\rho(w)}{\sin^2\left(\frac{2\pi t(x-w)}{a}\right)} - \frac{8\pi^2}{a}\,, \tag{4.17}$$

which is easily seen to be negative in the interval under consideration. This means that $G'(x)$ is a continuous decreasing function in the interval $|x| \geq \frac{a}{2\pi\, t} \arccos(\mathrm{e}^{-t^2/2a})$. Since $G'(x)$ vanishes at the endpoints of the interval, it must be negative. $G(x)$ is therefore a decreasing function for $|x| \geq \frac{a}{2\pi\, t} \arccos(\mathrm{e}^{-t^2/2a})$.

We conclude that if $x = \frac{1}{2} > \frac{a}{2\pi\, t} \arccos(e^{-t^2/2a})$ then

$$G(0) = G\left(\frac{a}{2\pi\, t} \arccos(\mathrm{e}^{-t^2/2a})\right) > G\left(\frac{1}{2}\right), \tag{4.18}$$

which implies that the zero-instanton contribution is exponentially larger than the one-instanton contribution. Instead, if $x = \frac{1}{2} \leq \frac{a}{2\pi\, t} \arccos(\mathrm{e}^{-t^2/2a})$ then

$$G(0) = G\left(\frac{1}{2}\right) \tag{4.19}$$

and from (4.15) we see that the two contributions are of the same order. We conclude that a critical curve exists and is given by

$$\frac{1}{2} = \frac{a}{2\pi\, t} \arccos(\mathrm{e}^{-t^2/2a}), \tag{4.20}$$

which separates two different regimes of the theory. In terms of the original parameters (t, p) it can be written in the form

$$t = p\, \log\left(\sec\left(\frac{\pi}{p}\right)^2\right). \tag{4.21}$$

In particular, we see that a critical value $t_c = p\, \log(\sec(\frac{\pi}{p})^2)$ is present at fixed p, with the instanton contributions to the free energy being exponentially suppressed for $t < t_c$.

4.2 Resurrecting the resolved conifold

In ordinary Yang-Mills on the sphere the analogous computations single out a critical value for the effective 't Hooft coupling $\lambda_c = \pi^2$. The statistical weight of instantons grows large enough above λ_c to make them the favorable configurations. The first natural check of our result (4.21) is to recover λ_c in the limit $p \to \infty$. Multiplying eq. (4.21) by p and recalling that the ordinary gauge theory is reached by sending $p \to \infty$ with $N\, g_s p = a$ fixed, we obtain $\lambda_c = \pi^2$ immediately.

The presence of λ_c separates the strong-coupling phase wherein instantons are not suppressed and a non-trivial master field is generated, from a weak-coupling phase governed by a gaussian master field. From the string theoretical point of view, the Gross-Taylor string expansion accurately describes the strong-coupling phase, while in weak-coupling phase stringy degrees of freedom do not appear. This suggests that, in a certain sense, strings emerge from instantons.

In the present case, for $t < t_c$, our result seems to support again the conclusion that the theory truncates to the zero-instanton sector in the large N-limit. The novelty this time is that the weak-coupling phase still admits a string description, but not the expected one! Let us consider the large N limit of the zero-instanton sector. Inserting $w_q^{\mathrm{inst}}(0, \dots, 0)$ into the instanton expansion we obtain

$$\mathcal{Z}^{0-\mathrm{inst}} = \frac{1}{N!}\left(\frac{2\pi}{g_s p}\right)^N \mathrm{e}^{-\frac{g_s(N^3-N)}{6p}} \int \prod_{i=1}^N \mathrm{d}z_i\, \mathrm{e}^{-\frac{8\pi^2 N}{a}\sum_{i=1}^N z_i^2} \prod_{1\le i<j\le N} \sin^2\left(\frac{2\pi t(z_i - z_j)}{a}\right). \tag{4.22}$$

As expected, the partition function (4.22) coincides with the partition function Z^p_{CS} of Chern-Simons theory on the Lens space L_p in the background of the trivial flat-connection with partition $\{N_k\} = (0, \dots, 0)$. According to [58, 59], eq. (4.22) is its matrix-model representation. The large N limit in the trivial vacuum is much easier to handle than in the general case. It can be explicitly performed by using, for example, the orthogonal polynomial technique explained in [68] or it can be obtained directly from the knowledge of the analogous computation for S^3 [69] by simply identifying the parameters. The partition function of the closed topological string theory on the resolved conifold $\mathcal{O}(1)\oplus\mathcal{O}(-1) \to \mathbb{P}^1$ emerges in the limit as an effect of the geometric transition. The only subtle point is that the string coupling constant, appearing in the closed string expansion, is renormalized as

$$\hat{g}_s = \frac{g_s}{p}\,. \tag{4.23}$$

This is expected from the relation with Chern-Simons theory, since the Kähler modulus is $\hat{t} = \hat{g}_s N$. This conclusion for the weak-coupling phase of q-deformed theory should be true at all order in $1/N$, the perturbative contribution to this expansion coming solely from the zero-instanton sector. In order to check this explicitly, an analysis in terms of discretized orthogonal polynomials, as done in ([25]) in the ordinary case, should be performed.

From the point of view of the conjecture [1], we have the rather surprising result that below a certain value of the Kähler modulus the string theory associated to q-deformed Yang-Mills theory does not seem to be the correct one. In particular, the background geometry does not depend on the parameter p. The Gopakumar-Vafa invariants, representing the BPS content of the theory, are the same as the ones of the resolved conifold.

One should expect that above the transition point $t > t_c$ the non-perturbative contributions, associated to the non-trivial flat connections, will generate the desired geometry.

JHEP01(2006)035

Looking closer at the transition curve (4.20), we realize that the cases $p = 1$ and $p = 2$ are exceptional from this point of view. The instanton contribution dominates when

$$\frac{\pi}{p} \geq \arccos(\mathrm{e}^{-t/2p}). \tag{4.24}$$

For $p = 1$ there are no real solutions for t while the case $p = 2$ experiences the transition at $t_c = \infty$. This suggests that for $p \leq 2$ the theory always remains in the weak-coupling phase. This conclusion is not affected by the presence of the parameter θ. One can easily check that the introduction of a θ-angle does not change the density, controlling the zero-instanton large N limit, nor the the ratio between the nonperturbative and perturbative contributions. In the case $p = 1$ the correct geometry is nevertheless obtained, but at this level we do not see any chiral-antichiral factorization, nor any fiber D-brane contributions. It is instructive to see explicitly how the behaviour changes above the value $p = 2$ by plotting the integral defining F_0 in eq. (4.13) (figure 1). In the subsequent sections we will

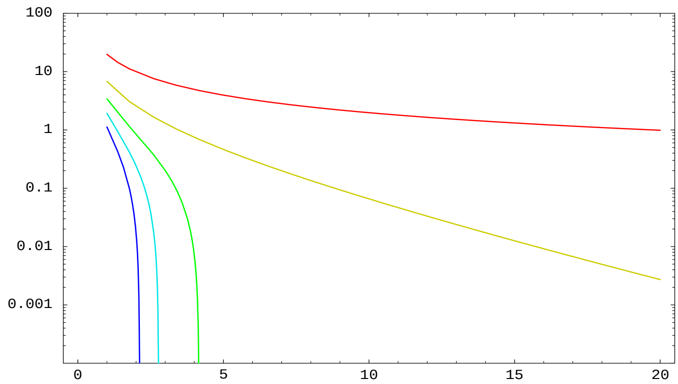

Figure 1: Plotting F_0 for $p = 1, 2, 3, 4, 5$ as a function of t

recover and extend these results by turning to a powerful saddle-point analysis, originally applied to ordinary Yang-Mills theory in [24].

5. Saddle point equation

An explicit result for the leading order (planar) contribution to the free energy of ordinary Yang-Mills theory on the sphere was obtained in [24]. For large area it fits nicely with the interpretation in terms of branched coverings that arises in the Gross-Taylor expansion, down to the phase transition point at $\lambda_c = \pi^2$ where the string series is divergent. We will now perform similar computations for q-deformed Yang-Mills theory on S^2.

5.1 Deformed Douglas-Kazakov phase transition

Recall that the partition function of the q-deformed gauge theory on S^2 is given by

$$Z^q_{\mathrm{YM}}(g_s, p) = \sum_{\substack{n_1,\ldots,n_N \in \mathbb{Z} \\ n_i - n_j \geq i-j \text{ for } i \geq j}} \mathrm{e}^{-\frac{g_s p}{2}(n_1^2 + \cdots + n_N^2)} \prod_{1 \leq i < j \leq N} \sinh^2\left(\frac{g_s}{2}(n_i - n_j)\right). \tag{5.1}$$

The constraint on the sums keeps track of the meaning of the integers n_i in terms of Young tableaux labels and highest weights. As in the previous section, we shall introduce the 't Hooft parameter $t = g_s N$ which is kept fixed as $N \to \infty$, and the additional parameter $a = g_s N\, p = t\, p$. In terms of these new variables, the partition function at $N \to \infty$ is given by

$$Z^q_{\rm YM}(t,a) = \exp\bigl(N^2 F(a,t)\bigr) = \lim_{N\to\infty} \sum_{\substack{n_1,\dots,n_N\in\mathbb{Z} \\ n_i-n_j\geq i-j \ \text{for}\ i\geq j}} \mathrm{e}^{\,-N^2 S_{\rm eff}(\vec{n})} \tag{5.2}$$

where

$$S_{\rm eff}(\vec{n}) = -\frac{1}{N^2}\sum_{i\neq j}\log\left[\sinh\left(\frac{t}{2}\,\frac{|n_i-n_j|}{N}\right)\right] + \frac{a}{2N}\sum_{i=1}^{N}\left(\frac{n_i}{N}\right)^2 . \tag{5.3}$$

The limit can be explicitly evaluated by means of saddle-point techniques usually employed in matrix models.

We introduce the variable $x_i = i/N$ and the function $n(x)$ such that

$$n(x_i) = \frac{n_i}{N} . \tag{5.4}$$

In the large N limit, x_i becomes a continuous variable $x \in [0,1]$ and $S_{\rm eff}$ can be written as an integral

$$S_{\rm eff} = -\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\ \log\left[\sinh\left(\frac{t}{2}\,\bigl|n(x)-n(y)\bigr|\right)\right] + \frac{a}{2}\int_0^1 \mathrm{d}x\ n(x)^2 . \tag{5.5}$$

In the continuum limit the constraint on the series (5.2) becomes

$$n(x) - n(y) \geq x - y \qquad \text{for} \qquad x \geq y . \tag{5.6}$$

From eq. (5.6) it follows that $n(x)$ is a monotonic function, and so we can consider its inverse $x(n)$. In eq. (5.5) we perform the change of variable $x \mapsto x(n)$ to get

$$S_{\rm eff} = -\int_{n(0)}^{n(1)} \mathrm{d}n \int_{n(0)}^{n(1)} \mathrm{d}n'\ \frac{\partial x(n)}{\partial n}\,\frac{\partial y(n')}{\partial n'}\ \log\left[\sinh\left(\frac{t}{2}\,|n-n'|\right)\right] + \frac{a}{2}\int_{n(0)}^{n(1)} \mathrm{d}n\ \frac{\partial x(n)}{\partial n}\ n^2 . \tag{5.7}$$

To simplify notation, we introduce the function

$$\rho(n) = \frac{\partial x(n)}{\partial n} \tag{5.8}$$

and denote the interval $[n(0), n(1)]$ by C. Then

$$S_{\rm eff} = -\int_C \mathrm{d}w \int_C \mathrm{d}w'\rho(w)\,\rho(w'\,)\ \log\left[\sinh\left(\frac{t}{2}\,|w-w'|\right)\right] + \frac{a}{2}\int_C \mathrm{d}w\ \rho(w)\ w^2 . \tag{5.9}$$

From eq. (5.6) we may translate the original constraint on the series in (5.2) in terms of the function ρ as

$$\rho(n) = \lim_{n'\to n} \frac{x(n') - x(n)}{n'-n} \leq 1 . \tag{5.10}$$

The constraint (5.10) is of fundamental importance in the calculation of the free energy, and in the underformed case [24] it leads to non-trivial consequences such as the large N phase transition and the existence of the strong-coupling phase. In the deformed case this same constraint holds as it follows from the fact that we started from a discrete summation over Young tableaux. The density $\rho(x)$ is positive since the function $x(n)$ is monotonic. Furthermore, one has

$$\int_C \mathrm{d}n\ \rho(n) = x\big(n(1)\big) - x\big(n(0)\big) = 1 - 0 = 1\,. \tag{5.11}$$

The distribution $\rho(x)$ in eq. (5.9) can be determined by requiring that it minimizes the action. This implies that it satisfies the saddle-point equation

$$\frac{a}{2}\,z = \frac{t}{2}\int_C \mathrm{d}w\ \rho(w)\ \coth\left(\frac{t}{2}\,(z-w)\right)\,. \tag{5.12}$$

This equation is a deformation of the usual Douglas-Kazakov equation that governs ordinary QCD_2 on the sphere and it is of the same type of eq. (4.8) which controls the zero-instanton sector. The ordinary gauge theory is recovered when $t \to 0$ while a is kept fixed according to the double-scaling limit. Eq. (5.12) then reduces to the usual saddle-point equation for the gaussian one-matrix model. The one-cut solution of eq. (5.12) is given in [68] where the large N limit of the Chern-Simons matrix model [68, 50] was considered.

Borrowing again the solution, the distribution $\rho(z)$ is given by

$$\rho(z) = \frac{a}{\pi\,t}\ \arctan\left(\sqrt{\frac{\mathrm{e}^{\,t^2/a}}{\cosh^2(\frac{t\,z}{2})} - 1}\ \right) \tag{5.13}$$

with the symmetric support

$$z \in \left[-\frac{2}{t}\,\mathrm{arccosh}(\,\mathrm{e}^{\,-t^2/2a})\ ,\ \frac{2}{t}\,\mathrm{arccosh}(\,\mathrm{e}^{\,-t^2/2a})\right]. \tag{5.14}$$

It can be considered as the q-deformation of the well-known Wigner semi-circle distribution. We now come to the crucial point. In order for $\rho(z)$ to be an acceptable solution, it has to satisfy the bound in eq. (5.10), i.e. it must be bounded from above by 1. We recall that this constraint originates from the structure of the discrete sum defining the partition function and it defines the regime in which the theory is well-approximated by a matrix model (up to nonperturbative $\frac{1}{N}$ corrections). To verify this constraint, we simply notice that the function $\rho(z)$ attains its absolute maximum at $z = 0$. In terms of the variables t and p, this produces the inequality

$$\arctan\left(\sqrt{\mathrm{e}^{\,t/p} - 1}\ \right) \leq \frac{\pi}{p}\,. \tag{5.15}$$

Since the $-\frac{\pi}{2} \leq \arctan(z) \leq \frac{\pi}{2}$, this condition is always satisfied for $p = 1$ or $p = 2$. The situation changes for $p > 2$. Since $\tan(z)$ is a monotonically increasing function for $z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, the inequality (5.15) can be equivalently written as $\sqrt{\mathrm{e}^{\,t/p} - 1} \leq \tan\frac{\pi}{p}$ which implies that

$$t \leq p\,\log\left(\sec^2\left(\frac{\pi}{p}\right)\right)\,. \tag{5.16}$$

This means that our solution breaks down when the 't Hooft coupling t reaches the critical value t_c found from the instanton computation in the previous section. This result reappears here from a different and more general point of view. Again, note that the cases $p = 1$ and $p = 2$ are special because then the one-cut solution is always valid. The relevant closed topological string theory is generated through a geometric transition, with the master field producing strings living on the resolved conifold.

The breakdown of the one-cut solution parallels exactly what happens in ordinary two-dimensional Yang-Mills theory, where it signals the appearance of a phase transition. In the saddle-point approach it is possible to go further and to find a solution describing the strong-coupling phase. To understand the nature of the phase transition and to describe the stringy geometry of the strong-coupling regime, we have to determine the distribution $\rho(z)$ that governs the system at $t > t_c$. Before proceeding with this analysis, we will compute the free energy in the weak-coupling phase.

5.2 Resolved conifold from the one-cut solution

The free energy in the weak-coupling phase is determined in terms of the gaussian potential and density $\rho(x)$ through

$$\mathcal{F}(t,a) = -\int_{-b}^{b} \mathrm{d}x \int_{-b}^{b} \mathrm{d}y \; \rho(x)\, \rho(y) \, \log \left| \sinh\left(\frac{t}{2}\,(x-y)\right)\right| + \frac{a}{2}\int_{-b}^{b} \mathrm{d}x \; x^2\, \rho(x)\,, \tag{5.17}$$

where the interval $C = [-b, b]$, $b = \frac{2}{t}\,\mathrm{arccosh}(\,\mathrm{e}^{\,t^2/2a})$ is the support of $\rho(x)$ given in eq. (5.14). Instead of working with eq. (5.17), it is simpler to compute its derivative with respect to a. In fact, when the derivative acts on one of the limits of the integrals it produces the distribution $\rho(x)$ evaluated at $x = b$ or $x = -b$, where it vanishes. Moreover, when the derivative is applied directly to $\rho(x)$ the different contributions cancel because of the saddle-point equation. The only remaining contribution is

$$\frac{\partial \mathcal{F}(t,a)}{\partial a} = \frac{1}{2}\int_{-b}^{b} \mathrm{d}x \; x^2\, \rho(x)\,, \tag{5.18}$$

as in the usual one-matrix model with a gaussian potential. There is, however, an important difference from the standard case. Normally, derivatives of the free energy are easily related to the Laurent expansion of the resolvent which solves the Riemann-Hilbert problem associated to the saddle-point equation. Here the presence of a deformed kernel (or of a non-polynomial potential in the matrix-model variables) means that no such simple relation exists, and the derivative of the free energy has to be computed by brute force techniques. This characteristic will complicate computations in the next section when we study two-cut solution. In the single-cut case, the final result is well-known [68] but it is instructive to perform the integral explicitly.

We have to evaluate the integral

$$\int_{-b}^{b} \mathrm{d}x \; x^2\, \rho(x) = \frac{a}{\pi\, t}\int_{-b}^{b} \mathrm{d}x \; x^2 \, \arctan\left(\sqrt{\frac{\mathrm{e}^{\,t^2/a}}{\cosh^2(\frac{t\,x}{2})} - 1}\;\right)\,. \tag{5.19}$$

Changing variables $x=(\log(y)-\frac{t^2}{a})/t$ we have

$$\int_{-b}^{b}\mathrm{d}x\ x^2\,\rho(x)=\frac{a}{\pi\,t^4}\int_{-\infty}^{\infty}\frac{\mathrm{d}y}{y}\ \left(\log(y)-\frac{t^2}{a}\right)^2\ \arctan\left(\frac{\sqrt{4y-(1+y\ \mathrm{e}^{-t^2/a})^2}}{1+y\ \mathrm{e}^{-t^2/a}}\right)$$
$$\times\,\Theta\left[4\,y-\left(1+y\ \mathrm{e}^{-t^2/a}\right)^2\right] \tag{5.20}$$

where $\Theta(y)$ is the step function. Starting from the integral representation

$$\rho\big(y(x);a,t\big)=\int_0^1\frac{\mathrm{d}\xi}{\pi\,t^2}\ \frac{\mathrm{e}^{-t^2\,\xi/a}\,\Theta\left[4\,y-\left(1+y\ \mathrm{e}^{-t^2\,\xi/a}\right)^2\right]}{\sqrt{4\,y-\left(1+y\ \mathrm{e}^{-t^2\,\xi/a}\right)^2}} \tag{5.21}$$

we may rewrite eq. (5.20) as

$$\int_{-b}^{b}\mathrm{d}x\ x^2\,\rho(x)=\int_0^1\frac{\mathrm{d}\xi}{\pi\,t^2}\ \int_{y_1(\xi)}^{y_2(\xi)}\mathrm{d}y\ \left(\log(y)-\frac{t^2}{a}\right)^2\ \frac{\mathrm{e}^{-t^2\,\xi/a}}{\sqrt{4\,y-\left(1+y\ \mathrm{e}^{-t^2\,\xi/a}\right)^2}}\ , \tag{5.22}$$

where y_1 and y_2 are the zeroes of the polynomial in the variable y which is the argument of the step function Θ in eq. (5.20), so that $[y_1(\xi),y_2(\xi)]$ is the support of Θ.

After another change of variable

$$y=-\,\mathrm{e}^{\,t^2\xi/a}+2\ \mathrm{e}^{\,2\,t^2\xi/a}+2\cos(\theta)\,\sqrt{-\,\mathrm{e}^{\,3\,t^2\xi/a}+\ \mathrm{e}^{\,4\,t^2\xi/a}} \tag{5.23}$$

the integral reduces to

$$\begin{aligned}\int_{-b}^{b}\mathrm{d}x\ x^2\,\rho(x)&=\int_0^1\frac{\mathrm{d}\xi}{a^2\pi\,t^2}\ \int_0^{\pi}\mathrm{d}\theta\ \Big(t^2\,(2\xi-1)\\&\qquad+a\,\log\left[2-\ \mathrm{e}^{-t^2\,\xi/a}+2\,\sqrt{1-\ \mathrm{e}^{-t^2\,\xi/a}}\ \cos\theta\right]\Big)^2\\&=\frac{t^2}{3a^2}+\frac{2}{t^2}\ \int_0^1\mathrm{d}\xi\ \mathrm{Li}_2\left(1-\ \mathrm{e}^{-t^2\,\xi/a}\right)\ ,\end{aligned} \tag{5.24}$$

where $\mathrm{Li}_n(x)$ is the polylogarithm function of order n. The remaining integral can be evaluated by integration by parts and all the remaining integrals are easily computed by employing the Mellin representation of the polylogarithm function to get

$$\begin{aligned}\int_{-b}^{b}\mathrm{d}x\ x^2\,\rho(x)&=\frac{t^2}{3a^2}+\frac{2}{t^2}\ \mathrm{Li}_2\left(1-\ \mathrm{e}^{-t^2/a}\right)-\frac{4a}{t^4}\,\zeta(3)-\frac{2}{a}\ \log\left(1-\ \mathrm{e}^{-t^2/a}\right)\\&\quad+\frac{4}{t^2}\ \mathrm{Li}_2\left(\mathrm{e}^{-t^2/a}\right)+\frac{4a}{t^4}\ \mathrm{Li}_3\left(\mathrm{e}^{-t^2/a}\right)\ .\end{aligned} \tag{5.25}$$

Next we use the polylogarithm identity

$$-\frac{\pi^2}{6}-x\ \log(1-\ \mathrm{e}^{-x})+\mathrm{Li}_2(\mathrm{e}^{-x})+\mathrm{Li}_2(1-\ \mathrm{e}^{-x})=0 \tag{5.26}$$

to simplify the integral finally to

$$\int_{-b}^{b}\mathrm{d}x\ x^2\,\rho(x)=\frac{t^2}{3a^2}+\frac{\pi^2}{3t^2}-\frac{4a}{t^4}\,\zeta(3)+\frac{2}{t^2}\ \mathrm{Li}_2(\mathrm{e}^{-t^2/a})+\frac{4a}{t^4}\ \mathrm{Li}_3(\mathrm{e}^{-t^2/a})\ . \tag{5.27}$$

This expression is easily integrated to give the free energy

$$\mathcal{F}(t,a) = -\frac{t^2}{6a} + \frac{\pi^2 a}{6t^2} - \frac{a^2}{t^4}\zeta(3) + \frac{a^2}{t^4}\,\mathrm{Li}_3(\mathrm{e}^{-t^2/a}) + c(t)\,, \tag{5.28}$$

which as expected coincides with the genus 0 free-energy of closed topological string theory on the resolved conifold. The a-independent function $c(t)$ can be easily determined by looking at the asymptotic expansion in t. In the limit $t \to 0$, $p \to \infty$ the free-energy of ordinary Yang-Mills theory in the weak-coupling phase is easily recovered. We have also to remark that the free energy depends only on the combination t^2/a (or $t_s/(p(p-2))$ in string variable). This dependence is in contrast with the one expected from the geometry of the Calabi-Yau $X = \mathcal{O}(p+2g-2) \oplus \mathcal{O}(-p) \longrightarrow \Sigma_g$. This picture, in fact, contains as a necessary ingredient two independent moduli e^{-t_s} and $e^{-\frac{2t_s}{p-2}}$.

6. Two-cut solution and the strong-coupling phase

For $p > 2$, the one-cut solution constructed earlier breaks down when the 't Hooft coupling constant t reaches the critical value t_c. To capture the behaviour of the theory above this threshold, we have to look for an ansatz that keeps track of the physical bound (5.10) satisfied by the distribution $\rho(x)$. We expect that the free energy obtained in this regime will reproduce the black hole partition function described in section 2 in terms of topological strings on $X = \mathcal{O}(-p) \oplus \mathcal{O}(p-2) \to \mathbb{P}^1$, with its chiral-antichiral dynamics and fiber D-brane contributions.

6.1 The two-cut solution

For $t > t_c$ we still have to solve the saddle point equation, but in the presence of the boundary condition (5.10). The new feature which may arise is that a finite fraction of Young tableaux variables $n_1, \ldots, n_N$ condense at the boundary of the inequality by respecting the parity symmetry of the problem,

$$n_{k+1} = n_{k+2} = \cdots = n_{N-k} = 0\,, \tag{6.1}$$

while all others are non-zero. This observation translates into a simple choice for the profile of $\rho(z)$ as depicted in figure 2. In order to respect the bound, the distribution function is chosen constant and equal to 1 everywhere in the interval $[-c, c]$. In the intervals $[-d, -c]$ and $[c, d]$ its form is instead dynamically determined by the saddle-point equation. Since the potential is symmetric under parity $z \to -z$, so is the function $\rho(z)$. The choice of symmetric ρ has a deeper meaning at the level of q-deformed Yang-Mills theory. It hides the assumption that the two-cut solution is dominated by the sector with vanishing U(1) charge. The asymmetry of the distribution can be related to this charge. This has been verified explicitly in the context of ordinary two-dimensional QCD in [54, 72].

Let us denote the set $[-d, -c] \cup [c, d]$ by $\mathcal{R}$ and the restriction of the distribution function to this set by $\tilde{\rho} = \rho|_{\mathcal{R}}$. Then the saddle-point equation

$$\frac{a}{2}\, z = \frac{t}{2} \int_{-d}^{d} \mathrm{d}w\ \rho(w)\, \coth\left(\frac{t}{2}\,(z-w)\right) \tag{6.2}$$

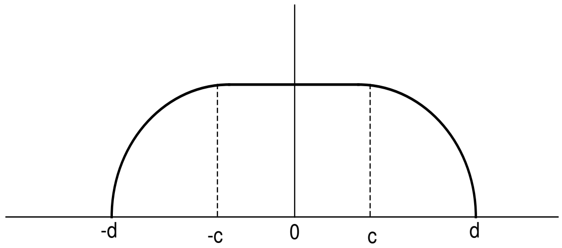

Figure 2: The double cut ansatz for the distribution $\rho(z)$.

can be rewritten as

$$\frac{a}{2}\,z - \log\left|\frac{\sinh\left(\frac{t}{2}\,(z+c)\right)}{\sinh\left(\frac{t}{2}\,(z-c)\right)}\right| = \frac{t}{2}\int_{\mathcal{R}} \mathrm{d}w\;\tilde{\rho}(w)\,\coth\left(\frac{t}{2}\,(z-w)\right). \tag{6.3}$$

It can be cast in a more standard form if we perform the change of variables $y = \mathrm{e}^{\,t\,z}\;\mathrm{e}^{\,t^2/a}$ and $u = \mathrm{e}^{\,t\,w}\;\mathrm{e}^{\,t^2/a}$. The saddle-point equation can then be written as

$$\frac{a}{2t^2}\,\frac{\log(y)}{y} - \frac{1}{t\,y}\,\log\left(\frac{\mathrm{e}^{\,c\,t-t^2/a}\,y-1}{\mathrm{e}^{\,-c\,t-t^2/a}\,y-1}\right) = \int_{\mathcal{R}} \mathrm{d}u\;\frac{\hat{\rho}(u)}{y-u}\,, \tag{6.4}$$

where we have defined

$$\hat{\rho}(u) = \frac{\tilde{\rho}\big(\log(u\;\mathrm{e}^{\,-t^2/a})/t\big)}{u\,t}\,. \tag{6.5}$$

The support $\mathcal{R}$ in terms of these new variables is given by

$$\mathcal{R} = [\,\underbrace{\mathrm{e}^{\,-t\,d+t^2/a}}_{\mathrm{e}^{\,d_-}}\,,\,\underbrace{\mathrm{e}^{\,-t\,c+t^2/a}}_{\mathrm{e}^{\,c_-}}\,]\;\cup\;[\,\underbrace{\mathrm{e}^{\,t\,c+t^2/a}}_{\mathrm{e}^{\,c_+}}\,,\,\underbrace{\mathrm{e}^{\,t\,d+t^2/a}}_{\mathrm{e}^{\,d_+}}\,]\,. \tag{6.6}$$

The original potential has been modified by a new term that depends explicitly on the endpoints of the support of the distribution. This apparently mild modification of the original one-cut problem will lead to a number of consequences. The original symmetry of our ansatz seems to have completely disappeared when the saddle-point equation is written in this way. This equation actually exhibits a more subtle symmetry. Let us consider the transformation

$$y \longmapsto \frac{\mathrm{e}^{\,2t^2/a}}{y} \tag{6.7}$$

which is the version of the original parity symmetry $z \mapsto -z$ in the new variables. It is straightforward to check that the saddle point equation and the region of integration are unaltered by (6.7). This invariance will be very important in the following, since it will govern many of the mysterious cancelations that will occur. In particular, it will play an important role in solving the apparent puzzle of having more equations than unknowns. At the level of the distribution function $\hat{\rho}(y)$ and of the corresponding resolvent $\omega(z)$, it implies the two useful functional relations

$$\hat{\rho}(y) = \frac{\mathrm{e}^{\,2t^2/a}}{y^2}\,\hat{\rho}\left(\frac{\mathrm{e}^{\,2t^2/a}}{y}\right) \tag{6.8}$$

and

$$\omega(z) = \int_{\mathcal{R}} \mathrm{d}w\;\frac{\hat{\rho}(w)}{w-z} = \frac{1}{z}\int_{\mathcal{R}} \mathrm{d}y\;\frac{y\,\hat{\rho}(y)}{\frac{\mathrm{e}^{\,2t^2/a}}{z}-y} = -\frac{1-2c}{z} - \frac{\mathrm{e}^{\,2t^2/a}}{z^2}\int_{\mathcal{R}} \mathrm{d}y\;\frac{\hat{\rho}(y)}{y-\frac{\mathrm{e}^{\,2t^2/a}}{z}}\;, \tag{6.9}$$

which imply that

$$\omega(z) + \frac{\mathrm{e}^{\,2t^2/a}}{z^2}\,\omega\left(\frac{\mathrm{e}^{\,2t^2/a}}{z}\right) = -\frac{1-2c}{z}\,. \tag{6.10}$$

The functional relation (6.10) states that the behaviour of the resolvent $\omega(z)$ at large z is determined by its behaviour about the origin. The only new information at $z \to \infty$ which is not present in the expansion around $z = 0$ is contained in the first term of the Laurent expansion and is given by

$$\omega(z) \simeq -\frac{1-2c}{z} \qquad \text{as} \quad z \to \infty\,. \tag{6.11}$$

It is exactly this term that carries the dynamical information. Imposing this condition will lead to two equations that determine the endpoints of the support intervals. It may seem puzzling that we are trying to fix four variables $(c_\pm, d_\pm)$ with only two equations, but by definition one has

$$c_+ + c_- = \frac{2t^2}{a}\,, \qquad d_- + d_+ = \frac{2t^2}{a}\,. \tag{6.12}$$

If we choose the cuts of the square root and of the logarithms as indicated in figure 3, then

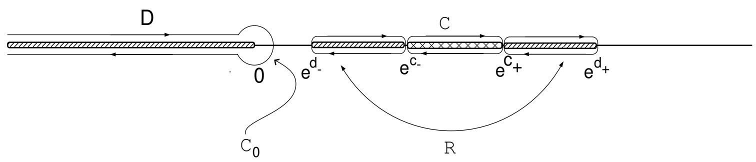

Figure 3: The cut structure of the distribution $\rho(z)$.

we can write the solution of the integral equation (6.3) as a contour integral around the cuts $[\,\mathrm{e}^{\,d_-},\,\mathrm{e}^{\,c_-}]$ and $[\,\mathrm{e}^{\,d_+},\,\mathrm{e}^{\,c_+}]$ as

$$\omega(z) = -\frac{1}{2\pi\,\mathrm{i}}\oint_{\mathcal{R}} \mathrm{d}w\;\frac{\frac{a}{2t^2}\,\frac{\log(w)}{w} - \frac{1}{t\,w}\,\log\left(\frac{\mathrm{e}^{\,-c_-}\,w-1}{\mathrm{e}^{\,-c_+}\,w-1}\right)}{z-w}$$

$$\times \sqrt{\frac{(z-\mathrm{e}^{c_-})(z-\mathrm{e}^{c_+})(z-\mathrm{e}^{d_+})(z-\mathrm{e}^{d_-})}{(w-\mathrm{e}^{c_-})(w-\mathrm{e}^{c_+})(w-\mathrm{e}^{d_+})(w-\mathrm{e}^{d_-})}}\,. \tag{6.13}$$

Since this integral decays as $\frac{1}{w^3}$ at infinity, we can deform the contour of integration so that it encircles the cuts of the two logarithms. This costs an additional contribution coming from the pole at $w = z$ and we find

$$\begin{aligned}
\omega(z) = & -\frac{a}{2t^2 z}\underbrace{\int_{-\infty}^{0}\frac{\mathrm{d}w}{z-w}\sqrt{\frac{(z-\mathrm{e}^{c_-})(z-\mathrm{e}^{c_+})(z-\mathrm{e}^{d_+})(z-\mathrm{e}^{d_-})}{(w-\mathrm{e}^{c_-})(w-\mathrm{e}^{c_+})(w-\mathrm{e}^{d_+})(w-\mathrm{e}^{d_-})}}}_{\mathcal{I}} \\
& -\frac{a}{2t^2 z}\underbrace{\int_{-\infty}^{-\epsilon}\frac{\mathrm{d}w}{w}\sqrt{\frac{(z-\mathrm{e}^{c_-})(z-\mathrm{e}^{c_+})(z-\mathrm{e}^{d_+})(z-\mathrm{e}^{d_-})}{(w-\mathrm{e}^{c_-})(w-\mathrm{e}^{c_+})(w-\mathrm{e}^{d_+})(w-\mathrm{e}^{d_-})}}}_{\mathcal{C}_0} \\
& \underbrace{+\frac{1}{t}\int_{\mathrm{e}^{c_-}}^{\mathrm{e}^{c_+}}\frac{\mathrm{d}w}{w(z-w)}\sqrt{\frac{(z-\mathrm{e}^{c_-})(z-\mathrm{e}^{c_+})(z-\mathrm{e}^{d_+})(z-\mathrm{e}^{d_-})}{(w-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-w)(\mathrm{e}^{d_+}-w)(w-\mathrm{e}^{d_-})}}}_{\mathcal{H}} \\
& \underbrace{-\frac{a}{2t^2}\frac{\log(z)}{z}}_{\mathcal{I}} + \underbrace{\frac{a\ \mathrm{e}^{-2t^2/a}\log(\epsilon)}{2\,t^2\,z}\sqrt{(z-\mathrm{e}^{c_-})(z-\mathrm{e}^{c_+})(z-\mathrm{e}^{d_+})(z-\mathrm{e}^{d_-})}}_{\mathcal{C}_0} \\
& +\frac{1}{t\,z}\log\left(\frac{\mathrm{e}^{-c_-}z-1}{\mathrm{e}^{-c_+}z-1}\right).
\end{aligned} \tag{6.14}$$

In order to determine the endpoints of the intervals, we have to impose the boundary condition (6.11). Recovering the expansion around $z = \infty$ is quite subtle and uninformative, and we describe it in detail in appendix B. We also show there that the spurious cut due to the presence of the logarithm disappears and a nice expansion to all orders can be given. The symbol $\mathcal{I}$ above represents the first integral together with the $\log(z)$ term, while $\mathcal{C}_0$ is the combination of the second integral and of the $\log(\epsilon)$ term. Both combinations are divided by the factor $\sqrt{(z-\mathrm{e}^{c_-})(z-\mathrm{e}^{c_+})(z-\mathrm{e}^{d_+})(z-\mathrm{e}^{d_-})}$, and similarly for the integral denoted $\mathcal{H}$. For $\mathcal{I}$, we have the expansion

$$\mathcal{I} = \frac{\mathsf{B}_0}{z} - \frac{\mathsf{B}_1}{z^2} + \cdots \tag{6.15}$$

and for the integral $\mathcal{H}$ we can write

$$\mathcal{H} = \frac{\mathcal{H}_{-1}}{z} + \frac{\mathcal{H}_0}{z^2} + \frac{\mathcal{H}_{-1}}{z^3} + \cdots\,. \tag{6.16}$$

The definitions of the integrals B_i, $\mathcal{C}_0$ and $\mathcal{H}_i$ are given in appendix B. Combining these expansions, we can write down a symbolic series for the resolvent up to order $\frac{1}{z}$ as

$$\begin{aligned}
\omega(z) \simeq & -\frac{a}{2t^2}\left[z\,\mathcal{C}_0 - \mathcal{C}_0 S_1 + \frac{S_2}{z}\mathcal{C}_0 + \mathsf{B}_0 - \frac{1}{z}\,(\mathsf{B}_1 + \mathsf{B}_0 S_1)\right] \\
& +\frac{1}{t}\left[\mathcal{H}_{-1}z + \mathcal{H}_0 - S_1\mathcal{H}_{-1} + \frac{1}{z}\,(S_2\mathcal{H}_{-1} - S_1\mathcal{H}_0 + \mathcal{H}_1)\right] + \frac{c_+ - c_-}{t\,z}\,,
\end{aligned} \tag{6.17}$$

where the coefficients S_i are defined through the asymptotic expansion

$$\frac{\sqrt{(z-\mathrm{e}^{c_-})(z-\mathrm{e}^{c_+})(z-\mathrm{e}^{d_+})(z-\mathrm{e}^{d_-})}}{z} = z - S_1 + \frac{S_2}{z} + \cdots . \tag{6.18}$$

Imposing the boundary condition (6.11) thereby produces three equations

$$\begin{aligned}
(z):&\qquad \frac{a}{2t}\,\mathcal{C}_0 - \mathcal{H}_{-1} = 0\,,\\
(z^0):&\qquad \frac{a}{2t}\,(-S_1\mathcal{C}_0 + \mathsf{B}_0) - \mathcal{H}_0 + S_1\mathcal{H}_{-1} = 0\,,\\
(z^{-1}):&\qquad -\frac{a}{2t}\,(S_2\mathcal{C}_0 - \mathsf{B}_1 - \mathsf{B}_0 S_1) + S_2\mathcal{H}_{-1} - S_1\mathcal{H}_0 + \mathcal{H}_1 + c_+ - c_- = -t(1-2c)
\end{aligned} \tag{6.19}$$

which combine to give

$$\begin{aligned}
\frac{a}{2t}\,\mathcal{C}_0 - \mathcal{H}_{-1} &= 0\,,\\
\frac{a}{2t}\,\mathsf{B}_0 - \mathcal{H}_0 &= 0\,,\\
\frac{a}{2t}\,\mathsf{B}_1 + \mathcal{H}_1 + 2t\,c &= -t(1-2c)\,.
\end{aligned} \tag{6.20}$$

There are three equations in two unknowns, but they are not independent. As shown in appendix B, the symmetry (6.7) implies the two further relations

$$\left(\mathsf{B}_1 + \frac{2t^2}{a}\right) = -\,\mathrm{e}^{2t^2/a}\,\mathcal{C}_0\,, \qquad \mathcal{H}_1 = \mathrm{e}^{2t^2/a}\,\mathcal{H}_{-1}\,. \tag{6.21}$$

Using these relations it is easy to show that the third equation in eq. (6.20) implies the first one. It suffices therefore consider only the first two equations, whose explicit integral forms are given respectively by

$$\begin{aligned}
&\frac{a}{2t}\int_0^\infty \mathrm{d}w\;\log(w)\,\frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{1}{\sqrt{(w+\mathrm{e}^{c_-})(w+\mathrm{e}^{c_+})(w+\mathrm{e}^{d_+})(w+\mathrm{e}^{d_-})}}\right)\\
&\qquad = \int_{\mathrm{e}^{c_-}}^{\mathrm{e}^{c_+}} \frac{\mathrm{d}w}{w}\,\frac{1}{\sqrt{(w-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-w)(\mathrm{e}^{d_+}-w)(w-\mathrm{e}^{d_-})}}\,,
\end{aligned} \tag{6.22}$$

$$\begin{aligned}
&\frac{a}{2t}\int_0^\infty \mathrm{d}w\;\frac{1}{\sqrt{(w+\mathrm{e}^{c_-})(w+\mathrm{e}^{c_+})(w+\mathrm{e}^{d_+})(w+\mathrm{e}^{d_-})}}\\
&\qquad = \int_{\mathrm{e}^{c_-}}^{\mathrm{e}^{c_+}} \frac{\mathrm{d}w}{\sqrt{(w-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-w)(\mathrm{e}^{d_+}-w)(w-\mathrm{e}^{d_-})}}\,.
\end{aligned} \tag{6.23}$$

We expect our two-cut solution to follow closely what happens in ordinary Yang-Mills theory at large N. When the two cuts merge ($c=0$), we should recover the one-cut values for the endpoints at the phase transition point. The equations above for the endpoints at $c=0$ reduce to

$$\pi\,\frac{\mathrm{e}^{t\,(d-\frac{4\,t}{a})/2}}{\mathrm{e}^{d\,t}-1} = -\frac{a\,\log\left(4\;\mathrm{e}^{2\,t^2/a}\right)}{2\;\mathrm{e}^{t^2/a}\,t} + \frac{a\,\log\left(\mathrm{e}^{-d\,t+t^2/a} + \mathrm{e}^{d\,t+t^2/a} + 2\;\mathrm{e}^{t^2/a}\right)}{2\;\mathrm{e}^{\frac{t^2}{a}}\,t}$$

$$+\frac{a\ \mathrm{e}^{\,t\,(d-\frac{4\,t}{a})/2}\ \arcsin\big(\tanh(\frac{d\,t}{2})\big)}{t-\ \mathrm{e}^{\,d\,t}\,t} \tag{6.24}$$

and

$$\pi\ \frac{\mathrm{e}^{\,t\,(d-\frac{2\,t}{a})/2}}{\mathrm{e}^{\,d\,t}-1}=\frac{a}{2t}\ \frac{2\ \mathrm{e}^{\,t\,(d-\frac{2\,t}{a})/2}\ \arcsin\big(\tanh(\frac{d\,t}{2})\big)}{\mathrm{e}^{\,d\,t}-1}\ , \tag{6.25}$$

where we have explicitly performed the integrals. Some simple algebra shows immediately that these two equations are equivalent to

$$d=\frac{2}{t}\,\mathrm{arccosh}\left(\mathrm{e}^{\,t^2/2a}\right)\qquad\text{and}\qquad t=-p\,\log\cos^2\left(\frac{\pi}{p}\right)\ . \tag{6.26}$$

These are exactly the conditions that determine endpoint b and the critical coupling constant in the one-cut solution.

6.2 The Douglas-Kazakov equations

We want to now understand how our equations are connected to the Douglas-Kazakov equations that arise in the large N limit of ordinary Yang-Mills theory on S^2. We can proceed easily once the integrals are performed in terms of elliptic functions (see appendix B). The equation (6.23) reads

$$\frac{a}{2t}\,\big(F(\nu_\infty,k)-F(\nu_0,k)\big)=\frac{a}{2t}\,F\left(\arcsin\left(\tanh\!\left(\frac{t\,(c+d)}{2}\right)\right),k\right)=K\,(k) \tag{6.27}$$

where

$$k=\sqrt{\frac{(\,\mathrm{e}^{\,c_+}-\,\mathrm{e}^{\,c_-})(\,\mathrm{e}^{\,d_+}-\,\mathrm{e}^{\,d_-})}{(\,\mathrm{e}^{\,d_+}-\,\mathrm{e}^{\,c_-})(\,\mathrm{e}^{\,c_+}-\,\mathrm{e}^{\,d_-})}} \tag{6.28}$$

and

$$\sin(\nu_0)=\sqrt{\frac{(\,\mathrm{e}^{\,d_+}-\,\mathrm{e}^{\,c_-})\ \mathrm{e}^{\,d_-}}{(\,\mathrm{e}^{\,d_+}-\,\mathrm{e}^{\,d_-})\ \mathrm{e}^{\,c_-}}}\,,\qquad \sin(\nu_\infty)=\sqrt{\frac{\mathrm{e}^{\,d_+}-\,\mathrm{e}^{\,c_-}}{\mathrm{e}^{\,d_+}-\,\mathrm{e}^{\,d_-}}}\,. \tag{6.29}$$

The symbols $K(k)=F(\pi/2,k)$ and $F(\nu,k)$ denote respectively the complete and incomplete elliptic integrals of the first kind. The intermediate equality in eq. (6.27) uses a well-known summation formula for elliptic integrals [70]. In the limit $t\to 0$, the small angle behaviour of F reduces eq. (6.27) to

$$\frac{a}{4}\,(c+d)=K(\hat{k}) \tag{6.30}$$

where $\hat{k}=2\sqrt{c\,d}/(c+d)=2\sqrt{\ell}/(1+\ell)$ and $\ell=c/d$ is the Douglas-Kazakov modulus. From the transformation property

$$K\left(\frac{2\sqrt{\ell}}{1+\ell}\right)=(1+\ell)K(\ell)=\frac{c+d}{d}\,K(\ell) \tag{6.31}$$

we recover

$$d=\frac{4}{a}\,K(\ell) \tag{6.32}$$

which is the first Douglas-Kazakov equation.

Let us now turn to eq. (6.22). The situation here is more involved since we have to deal with (incomplete and complete) elliptic integrals of the third kind $\Pi(\nu, n, k)$ and $\Pi(n, k) = \Pi(\pi/2, n, k)$, which do not arise in ordinary two-dimensional Yang-Mills theory. Evaluating all the integrals, we find

$$\begin{aligned}
&\frac{2\ \mathrm{e}^{-2t^2/a}}{\sqrt{(\mathrm{e}^{d_+}-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-\mathrm{e}^{d_-})}}\left[\left(\mathrm{e}^{c_-}-\mathrm{e}^{d_-}\right)\Pi\left(\frac{\mathrm{e}^{c_+}-\mathrm{e}^{c_-}}{\mathrm{e}^{c_+}-\mathrm{e}^{d_-}}, k\right)+\mathrm{e}^{d_-}\,K(k)\right]\\
&\quad= -\frac{a}{2t}\,\mathrm{e}^{-2t^2/a}\left[\frac{2}{\sqrt{(\mathrm{e}^{c_+}-\mathrm{e}^{d_-})(\mathrm{e}^{d_+}-\mathrm{e}^{c_-})}}\left\{\left(\mathrm{e}^{c_-}-\mathrm{e}^{d_-}\right)\left(\Pi\big(\nu_0, \frac{\mathrm{e}^{c_+}-\mathrm{e}^{c_-}}{\mathrm{e}^{c_+}-\mathrm{e}^{d_-}}, k\big)\right.\right.\right.\\
&\qquad\left.\left.-\ \Pi\big(\nu_\infty, \frac{\mathrm{e}^{c_+}-\mathrm{e}^{c_-}}{\mathrm{e}^{c_+}-\mathrm{e}^{d_-}}, k\big)\right)-\mathrm{e}^{d_-}\,\big(F(\nu_\infty, k)-F(\nu_0, k)\big)\right\}\\
&\qquad\left.-\log\left(\frac{1}{4}\,(\mathrm{e}^{d_-}+\mathrm{e}^{c_-})\,(\mathrm{e}^{-d_+-d_-}+\mathrm{e}^{-c_--d_-})\right)\right]\,.
\end{aligned}\tag{6.33}$$

With the help of eq. (6.23) we obtain

$$\begin{aligned}
\Pi\left(\frac{\mathrm{e}^{c_+}-\mathrm{e}^{c_-}}{\mathrm{e}^{c_+}-\mathrm{e}^{d_-}}, k\right) &= \frac{a}{2t}\left[\Pi\left(\nu_\infty, \frac{\mathrm{e}^{c_+}-\mathrm{e}^{c_-}}{\mathrm{e}^{c_+}-\mathrm{e}^{d_-}}, k\right)-\Pi\left(\nu_0, \frac{\mathrm{e}^{c_+}-\mathrm{e}^{c_-}}{\mathrm{e}^{c_+}-\mathrm{e}^{d_-}}, k\right)\right]\\
&\quad+\frac{a}{4t}\,\frac{\sqrt{(\mathrm{e}^{d_+}-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-\mathrm{e}^{d_-})}}{\mathrm{e}^{c_-}-\mathrm{e}^{d_-}}\\
&\quad\times\ \log\left(\frac{1}{4}\,(\mathrm{e}^{d_-}+\mathrm{e}^{c_-})(\mathrm{e}^{-d_+-d_-}+\mathrm{e}^{-c_--d_-})\right)\,.
\end{aligned}\tag{6.34}$$

To reach ordinary two-dimensional QCD, we expand around $t = 0$ with a fixed. At lowest order t^0 we find

$$\Pi\left(\frac{2c}{c+d}, \frac{2\sqrt{c\,d}}{c+d}\right) = a\,\frac{d\ (c+d)}{4\ (d-c)}\,.\tag{6.35}$$

Since

$$n = \frac{2c}{c+d} = 1-\sqrt{1-k^2}\tag{6.36}$$

the complete elliptic integral of the third kind reduces to one of the first kind due to the identity

$$\Pi\left(\frac{2c}{c+d}, \frac{2\sqrt{c\,d}}{c+d}\right) = \frac{d}{d-c}\,K\left(\frac{2\sqrt{c\,d}}{c+d}\right)\,.\tag{6.37}$$

We can therefore write the lowest order equation as

$$K\left(\frac{2\sqrt{c\,,d}}{c+d}\right) = a\,\frac{c+d}{4}\,,\tag{6.38}$$

which coincides with eq. (6.27) and hence the first Douglas-Kazakov equation.

To obtain the second equation, we expand eq. (6.34) to second order in t to get

$$\frac{(c+d)^2}{4\ (c-d)}\left[E\left(\frac{2\sqrt{c\,d}}{c+d}\right)-K\left(\frac{2\sqrt{c\,d}}{c+d}\right)\right] = -\frac{a\,c\,d\ (c+d)}{8\ (c-d)}+\frac{(c+d)\ \big(4-a\,(c^2+d^2)\big)}{16\ (c-d)}\tag{6.39}$$

where $E(k)$ is the complete elliptic integral of the second kind. Simplifying we get

$$(c+d)\,E\left(\frac{2\sqrt{c\,d}}{c+d}\right) = 1\,, \tag{6.40}$$

which finally yields the second Douglas-Kazakov equation after employing the modular property

$$(c+d)E\left(\frac{2\sqrt{c\,d}}{c+d}\right) = d\left[2E\left(\frac{c}{d}\right) - \left(1-\frac{c^2}{d^2}\right)K\left(\frac{c}{d}\right)\right]. \tag{6.41}$$

One can verify that expanding eq. (6.27) to the next order in t does not produce any new conditions because its expansion does not contain terms linear in t. The fact that the second equation appears only at second order in t should come as no surprise, as we have summed the equations with a weight depending on t in order to have the simplest possible expressions. This procedure mixes the various orders of the expansion.

6.3 Saddle-point solution and the transition curve

It is instructive to show that the double cut equations are solvable only for $p > 2$, in agreement with the result that the phase transition exists only in this region. This can be achieved in two ways. From the summation formula [70] we can easily show that

$$F(\nu_0,k) + F(\nu_\infty,k) = K(k)\,. \tag{6.42}$$

Then eq. (6.27) can be written in the form

$$(p-2)\,F(\nu_\infty,k) = (p+2)\,F(\nu_0,k) \tag{6.43}$$

where we have used $a = p\,t$. Since

$$\frac{\partial F(\nu,k)}{\partial\nu} = \frac{1}{\sqrt{1-k^2\sin^2(\nu)}} > 0 \tag{6.44}$$

the function F is real and monotonic for any k. Since $F(0,k) = 0$, it is positive for any $\nu > 0$ and $0 \le k < 1$. Since both ν_0 and ν_∞ are positive, $F(\nu_\infty,k)$ and $F(\nu_0,k)$ are also positive. The equation thereby admits solutions if and only if

$$p > 2\,. \tag{6.45}$$

Alternatively, eq. (6.27) can be solved by determining $(c+d)$ in terms of the modular parameter k. We have

$$c+d = \frac{2}{t}\,\mathrm{arctanh}\left(\mathrm{sn}\left(\frac{2t\,K(k)}{a},k\right)\right) = \frac{2}{t}\,\mathrm{arctanh}\left(\mathrm{sn}\left(\frac{2K(k)}{p},k\right)\right) \tag{6.46}$$

where $\mathrm{sn}(x,k)$ is the elliptic sine function of modular parameter k. From this representation one can explicitly check that no solution exists for $p = 1$ and $p = 2$. For $p = 1$ we have $\tanh(\frac{(c+d)t}{2}) = 0$ implying $c = d = 0$, since both c and d are non-negative. For $p = 2$ we find $\tanh(\frac{(c+d)t}{2}) = 1$ implying $t(c+d) = \infty$, and so in this case we can obtain finite values for c and d only if $t \to \infty$.

The appearance of elliptic functions suggests that the natural unknown parameter for our equations are not the endpoints (c, d) of the support interval, but rather the modulus k [24]. We can eliminate all explicit dependence on (c, d) through the condition (6.46) and

$$\tanh\left(\frac{(d-c)t}{2}\right) = k' \frac{\mathrm{sn}(x,k)}{\mathrm{dn}(x,k)} = -\mathrm{cn}\left(\frac{p+2}{p} K(k), k\right) \tag{6.47}$$

where $x = 2K(k)/p$, $k' = \sqrt{1-k^2}$ is the complementary modulus and $\mathrm{dn}(x,k)$ is the elliptic amplitude. For future convenience we shall also introduce the parameter $\hat{x} = (p+2)K(k)/p$. This second relation is the translation of the definition (6.28). Then we can write down a table of translations for our parameters as

$$n = \frac{\mathrm{e}^{\,c_+} - \mathrm{e}^{\,c_-}}{\mathrm{e}^{\,c_+} - \mathrm{e}^{\,d_-}} = k^2\, \mathrm{sn}^2\left(\frac{\hat{x}}{2}, k\right) \tag{6.48}$$

and

$$\sin(\nu_\infty) = \frac{\sqrt{n}}{k} = \mathrm{sn}\left(\frac{\hat{x}}{2}, k\right), \quad \sin(\nu_0) = \frac{\cos(\nu_\infty)}{\sqrt{1-k^2\sin^2(\nu_\infty)}} = \frac{\mathrm{cn}\left(\frac{\hat{x}}{2}, k\right)}{\mathrm{dn}\left(\frac{\hat{x}}{2}, k\right)}. \tag{6.49}$$

With this table of translations, eq. (6.34) can be written as

$$\frac{t}{4} = \frac{p}{4} \log\left(\frac{\mathrm{dn}(x,k)}{\mathrm{cn}^2(x,k)}\right) + \frac{p}{2} \frac{k' \,\mathrm{cn}(x,k)}{1+\mathrm{sn}(x,k)} \left[\Pi\left(\nu_\infty, n, k\right) - \Pi\left(\nu_0, n, k\right) - \frac{2}{p}\,\Pi\left(n, k\right)\right] \tag{6.50}$$

The graphical behaviour of the function on the right-hand side of eq. (6.50) is depicted in figure 4, and one can check that it is a monotonically increasing function as k runs from 0 to 1. Thus in order to have a solution, $\frac{t}{4}$ must be greater than the value of the right-hand

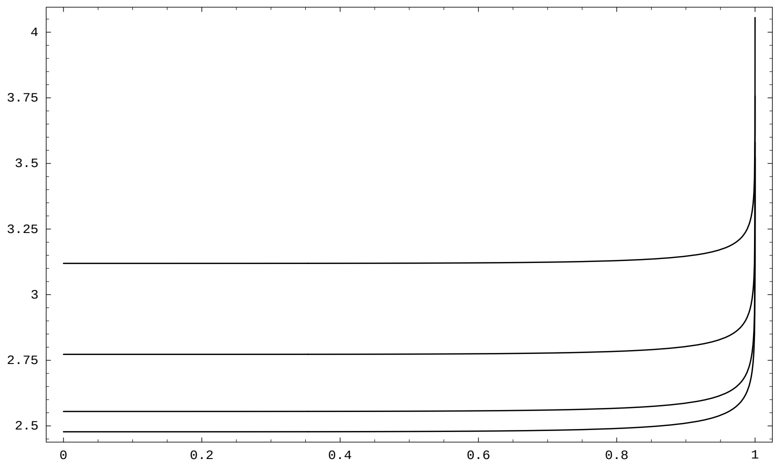

Figure 4: The right-hand side of the saddle-point equation times p for $p = 3, 4, 7, 20$.

side of eq. (6.50) at $k = 0$. For $k = 0$, only the first term in eq. (6.50) survives giving the bound

$$\frac{t}{4} \geq \frac{p}{4} \log \left(\frac{\mathrm{dn}\left(\frac{\pi}{p}, 0\right)}{\mathrm{cn}^2\left(\frac{\pi}{p}, 0\right)} \right) = \frac{p}{4} \log \left(\sec^2 \left(\frac{\pi}{p} \right) \right) \equiv \frac{t_c}{4} \,. \tag{6.51}$$

In other words, two-cut solutions exist only when t is above the transition curve determined by the one-cut analysis. The procedure to determine the solutions is now clear. Given (t, p) one solves eq. (6.50) for k. Since the right-hand side of eq. (6.50) is monotonic, the solution is unique. Then from eqs. (6.46) and (6.47) we obtain (c, d).

Eq. (6.50) is very complicated and not promising for an analytical treatment. The situation improves dramatically if we change variable from k to the modular parameter $q = \exp(-\pi\, K'(k)/K(k))$. Then the right-hand side of the equation reduces to an elegant q-series given by

$$\frac{t}{4} = \frac{t_c}{4} - 2p \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \, \frac{q^{2n}}{1 - q^{2n}} \, \sin^2 \left(\frac{\pi\, n}{p} \right) \tag{6.52}$$

as shown in appendix C. This series can also be summed in terms of theta-functions to get

$$\frac{t}{4} = -\frac{p}{2} \left[\log \left(\cos\left(\frac{\pi}{p}\right) \right) + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \, \frac{q^{2n}}{1 - q^{2n}} \, \sin^2 \left(\frac{\pi\, n}{p} \right) \right] = -\frac{p}{2} \log \left(\frac{\vartheta_2 \left(\frac{\pi}{p} \middle| \, q \right)}{\vartheta_2\,(0|\, q)} \right) \,. \tag{6.53}$$

In this form the connection with the DK equation determining the modulus q is very simple. Multiplying by p and taking the limit $p \to \infty$, at leading order we find

$$\frac{a}{4} = \frac{\pi^2}{4} - 2\pi^2 \sum_{n=1}^{\infty} (-1)^n \frac{n\, \hat{q}^{2n}}{1 - \hat{q}^{2n}} = \frac{\pi^2}{4} \left(1 + 8\, \hat{q}^2 - 8\, \hat{q}^4 + 32\, \hat{q}^6 - 40\, \hat{q}^8 + \ldots \right) \tag{6.54}$$

where $\hat{q} = \lim_{p \to \infty} q$. This is exactly the Douglas-Kazakov equation, except that it is a series in $\hat{q}^2$ rather than in $\hat{q}$. This apparent discrepancy can be easily resolved if we recall that our modulus k and consequently $\hat{q}$ are different from the Douglas-Kazakov moduli. One can show that

$$\hat{q} = \exp \left(-\pi \, \frac{K'(\hat{k})}{K(\hat{k})} \right) = \exp \left(-\frac{\pi}{2} \, \frac{K'\,(\ell)}{K\,(\ell)} \right) \tag{6.55}$$

and so our modular parameter is the square root of the Douglas-Kazakov modular parameter. We stress that the knowledge of q is equivalent to knowledge of k. By employing the q-expansions of the elliptic trigonometric functions, the endpoints (c, d) can also be expressed in terms of q. Explicit formulas are given in appendix C.

6.4 The third-order phase transition

Let us now investigate the behaviour of the theory near the critical point. It is possible to derive a perturbative solution of our equation just above the phase transition. At the critical

JHEP01(2006)035

point t_c, we have $q=0$. From eqs. (6.46) and (6.47) we find that $d\pm c = \frac{2}{t_c}\,\mathrm{arctanh}(\sin(\frac{\pi}{p}))$, which implies that $c=0$ and

$$d = \frac{2}{t_c}\,\mathrm{arctanh}\left(\sin\left(\frac{\pi}{p}\right)\right) = \frac{2}{t_c}\,\mathrm{arccosh}\left(\sec\left(\frac{\pi}{p}\right)\right) = \frac{2}{t_c}\,\mathrm{arccosh}\left(\mathrm{e}^{\,t_c^2/2a_c}\right). \tag{6.56}$$

At the critical point t_c the two cuts merge and form a single cut whose endpoints coincide with those of the one-cut solution. To go further and see what happens in a neighborhood around t_c we have to solve eq. (6.52) as a series in $(t-t_c)$. We assume that q admits an expansion of the form

$$q = (t-t_c)^{\alpha}\sum_{n=0}^{\infty} a_n (t-t_c)^n\,. \tag{6.57}$$

Substituting this expansion into eq. (6.52), we can iteratively solve for the coefficients a_n and find

$$\begin{aligned} q \,=\, \sqrt{\hat{t}}\,\Biggl(&\frac{\csc(\frac{\pi}{p})}{\sqrt{8p}} + \frac{\hat{t}\,\cos(\frac{2\pi}{p})\csc^3(\frac{\pi}{p})}{32\sqrt{2}\,p^{3/2}} + \frac{\hat{t}^2\left(-27-32\,\cos(\frac{2\pi}{p})+5\,\cos(\frac{4\pi}{p})\right)\csc^5(\frac{\pi}{p})}{6144\sqrt{2}\,p^{5/2}} \\ &+ \frac{\hat{t}^3\left(-64-157\,\cos(\frac{2\pi}{p})-64\,\cos(\frac{4\pi}{p})+\cos(\frac{6\pi}{p})\right)\csc^7(\frac{\pi}{p})}{65536\sqrt{2}\,p^{7/2}} + O(\hat{t}^4)\Biggr) \end{aligned} \tag{6.58}$$

where $\hat{t} = t-t_c$. From this expansion we can obtain all information about the gauge theory in the strong-coupling phase around the critical point.

To investigate the behaviour of the theory beyond the phase transition, we need to understand what happens to the distribution function ρ. Because our potential is non-polynomial, we have no simple relations relating derivatives of the free energy to the expansion of the resolvent, as in the standard matrix models or in ordinary Yang-Mills theory. We have to resort therefore to a brute force calculation. The first step is to compute the resolvent in a closed form in terms of elliptic functions. Using the first saddle-point equation (6.22) to simplify the expression (6.14), we find

$$\begin{aligned} \omega(z) \,=\, &-\frac{a}{2t^2 z}\int_0^{\infty}\frac{\mathrm{d}w}{z+w}\sqrt{\frac{(z-\mathrm{e}^{\,c_-})(z-\mathrm{e}^{\,c_+})(z-\mathrm{e}^{\,d_+})(z-\mathrm{e}^{\,d_-})}{(w+\mathrm{e}^{\,c_-})(w+\mathrm{e}^{\,c_+})(w+\mathrm{e}^{\,d_+})(w+\mathrm{e}^{\,d_-})}} \\ &+\frac{1}{t\,z}\int_{\mathrm{e}^{\,c_-}}^{\mathrm{e}^{\,c_+}}\frac{\mathrm{d}w}{z-w}\sqrt{\frac{(z-\mathrm{e}^{\,c_-})(z-\mathrm{e}^{\,c_+})(z-\mathrm{e}^{\,d_+})(z-\mathrm{e}^{\,d_-})}{(w-\mathrm{e}^{\,c_-})(\mathrm{e}^{\,c_+}-w)(\mathrm{e}^{\,d_+}-w)(w-\mathrm{e}^{\,d_-})}} \\ &-\frac{a}{2t^2}\,\frac{\log(z)}{z} + \frac{1}{t\,z}\,\log\left(\frac{\mathrm{e}^{\,-c_-}\,z-1}{\mathrm{e}^{\,-c_+}\,z-1}\right) \\ =\, &\frac{2\sqrt{(z-\mathrm{e}^{\,c_-})(z-\mathrm{e}^{\,c_+})(z-\mathrm{e}^{\,d_+})(z-\mathrm{e}^{\,d_-})}\,(\mathrm{e}^{\,c_-}-\mathrm{e}^{\,d_-})}{t\,z(z-\mathrm{e}^{\,c_-})(z-\mathrm{e}^{\,d_-})\sqrt{(\mathrm{e}^{\,d_+}-\mathrm{e}^{\,c_-})(\mathrm{e}^{\,c_+}-\mathrm{e}^{\,d_-})}} \\ &\times\left[\Pi\left(\frac{(\mathrm{e}^{\,c_+}-\mathrm{e}^{\,c_-})(z-\mathrm{e}^{\,d_-})}{(\mathrm{e}^{\,c_+}-\mathrm{e}^{\,d_-})(z-\mathrm{e}^{\,c_-})},k\right) + \frac{a}{2t}\left\{\Pi\left(\nu_\infty,\frac{(\mathrm{e}^{\,d_+}-\mathrm{e}^{\,d_-})(z-\mathrm{e}^{\,c_-})}{(\mathrm{e}^{\,d_+}-\mathrm{e}^{\,c_-})(z-\mathrm{e}^{\,d_-})},k\right)\right.\right. \\ &\left.\left.-\,\Pi\left(\nu_0,\frac{(\mathrm{e}^{\,d_+}-\mathrm{e}^{\,d_-})(z-\mathrm{e}^{\,c_-})}{(\mathrm{e}^{\,d_+}-\mathrm{e}^{\,c_-})(z-\mathrm{e}^{\,d_-})},k\right)\right\} - K(k)\right] \end{aligned} \tag{6.59}$$

$$-\frac{a}{2t^2}\,\frac{\log(z)}{z}+\frac{1}{t\,z}\,\log\left(\frac{\mathrm{e}^{-c_-}\,z-1}{\mathrm{e}^{-c_+}\,z-1}\right)\,. \tag{6.60}$$

From the discontinuities of this expression we can derive the distribution of Young tableaux variables in the various intervals. In the region $z\in[\,\mathrm{e}^{c_+},\,\mathrm{e}^{d_+}]$ we find

$$\begin{aligned}\rho &= \frac{1}{2\pi\,\mathrm{i}}\,\big(\omega(z+\mathrm{i}\,\epsilon)-\omega(z-\mathrm{i}\,\epsilon)\big)\\ &= \frac{2(\mathrm{e}^{c_-}-\mathrm{e}^{d_-})}{\pi\,t\,z\,\sqrt{(\mathrm{e}^{d_+}-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-\mathrm{e}^{d_-})}}\,\sqrt{\frac{(z-\mathrm{e}^{c_+})(\mathrm{e}^{d_+}-z)}{(z-\mathrm{e}^{d_-})(z-\mathrm{e}^{c_-})}}\\ &\times\left[\Pi\left(\frac{(\mathrm{e}^{c_+}-\mathrm{e}^{c_-})(z-\mathrm{e}^{d_-})}{(\mathrm{e}^{c_+}-\mathrm{e}^{d_-})(z-\mathrm{e}^{c_-})},k\right)+\frac{a}{2t}\,\left\{\Pi\left(\nu_\infty,\frac{(\mathrm{e}^{d_+}-\mathrm{e}^{d_-})(z-\mathrm{e}^{c_-})}{(\mathrm{e}^{d_+}-\mathrm{e}^{c_-})(z-\mathrm{e}^{d_-})},k\right)\right.\right.\\ &\left.\left.-\Pi\left(\nu_0,\frac{(\mathrm{e}^{d_+}-\mathrm{e}^{d_-})(z-\mathrm{e}^{c_-})}{(\mathrm{e}^{d_+}-\mathrm{e}^{c_-})(z-\mathrm{e}^{d_-})},k\right)\right\}-K(k)\right]\,. \end{aligned}\tag{6.61}$$

The explicit form of the distribution function in the dual region $z\in[\,\mathrm{e}^{d_-},\,\mathrm{e}^{c_-}]$ is obtained by means of the symmetry (6.7). The elliptic integrals of the third kind appearing in the expression for ρ have parameter n (the second entry of the function) in the interval $k^2<n<1$ when $z\in[\,\mathrm{e}^{c_+},\,\mathrm{e}^{d_+}]$. They are thus of circular type [75]. In what follows we shall use the q-expansion derived in appendix C to determine the behaviour of the distribution $\rho(z)$ near the critical point t_c.

One can write

$$\begin{aligned}\rho(z) &= \frac{2}{\pi\,t\,z}\left(\frac{\pi}{2}-\frac{\pi}{2}\,\Lambda(\eta,k)+\frac{p}{2}\,\arctan\left(\tan\left(\frac{\pi}{p}\right)\,\tanh\big(2\beta_1\big)\right)-2\pi\,\mu(\beta_1)\right.\\ &\left.-\sum_{s=1}^{\infty}\frac{p}{s}\,\frac{q^{4s}}{1-q^{4s}}\,\sin\left(\frac{2\pi\,s}{p}\right)\,\sinh\big(2s\,\beta_1\big)\right)\end{aligned}\tag{6.62}$$

where Λ is the Heuman lambda-function and μ is a ratio of theta-functions which is given explicitly in appendix C. The definitions of the parameters η and β_1 can also be found in appendix C. For the moment their explicit forms are not important. At $t=t_c$, the modular parameter q vanishes and the only term surviving in the expansion of ρ is the one involving the arctangent function. At the critical point the distribution is therefore given by

$$\rho_c(z)=\frac{p}{\pi\,t\,z}\,\arctan\left(\tan\left(\frac{\pi}{p}\right)\,\tanh\big(2\beta_1^c(z)\big)\right)\,. \tag{6.63}$$

In the original variables it takes the form

$$\rho_c(x)=\frac{p}{\pi}\,\arctan\left(\tan\left(\frac{\pi}{p}\right)\,\tanh\big(2\beta_1^c(\mathrm{e}^{t_c x+t_c^2/a_c})\big)\right)=\frac{p}{\pi}\,\arctan\left(\sqrt{\frac{\mathrm{e}^{t^2/a}}{\cosh^2(\frac{t\,x}{2})}-1}\right) \tag{6.64}$$

and thus coincides with the one-cut distribution. This result shows that the free energy and its first derivative are continuous at the critical point. The second derivative of the free energy can be reduced to computing

$$\frac{\partial^2\mathcal{F}}{\partial a^2}\propto\frac{1}{2}\int_{\mathrm{e}^{c_+}}^{\mathrm{e}^{d_+}}\mathrm{d}z\,\frac{\partial\rho(z)}{\partial a}\,\left(\log(z)-\frac{t^2}{a}\right)^2+\frac{1}{2}\int_{\mathrm{e}^{d_-}}^{\mathrm{e}^{c_-}}\mathrm{d}z\,\frac{\partial\rho(z)}{\partial a}\,\left(\log(z)-\frac{t^2}{a}\right)^2 \tag{6.65}$$

as all other contributions vanish because of the boundary conditions on the distribution and its symmetries. The derivatives are taken at constant t. A tedious expansion of this quantity around $t = t_c$ using *Mathematica* shows that it vanishes linearly in $(a - a_c)$ and thus the phase transition is of third order.

6.5 Evidence for the string picture

The complete discussion of the string picture, namely the identification of the Calabi-Yau geometry and the appearance of the perturbative topological string expansion, is out of the scope of the present paper whose intent is to focus more on field theoretical features. These string theoretic issues will be discussed in full detail in a forthcoming paper [28]. Nevertheless, we can present some evidences for how the topological string expansion emerges which also resolves an apparent puzzle. The topological string perturbative expansion is naturally organized as a double series in two modular parameters e^{-t_s} and $\mathrm{e}^{-2t_s/(p-2)}$, where t_s is the Kähler modulus (related to our t by $t = \frac{2t_s}{p-2}$). The appearance of this double dependence from our equation is non-trivial and absolutely necessary for the string interpretation. It is absent in the weak coupling region where everything can be written as a series in a single modulus given by $\mathrm{e}^{-t/2p} = \mathrm{e}^{-t_s/p(p-2)}$.

We expect that the topological string theory will arise when t is large. Thus we have to investigate the solution of our saddle-point equation around $t = \infty$. In this region, k and consequently q approach 1. If we employ the standard parametrization for the modular parameter $q = \mathrm{e}^{\pi\,\mathrm{i}\,\tau}$, then $\tau \to 0$. When studying the behaviour of an elliptic function as $\tau \to 0$, the natural technique is to perform a modular transformation. This procedure exchanges τ and $-\frac{1}{\tau}$, and thus takes us back to the weak-coupling phase where a perturbative solution can be attempted. Our saddle-point equation then becomes

$$\frac{t}{4} = -\frac{p}{2}\log\left(\frac{\vartheta_2\left(\left.\frac{\pi}{p}\right| q\right)}{\vartheta_2\left(0|\, q\right)}\right) = -\frac{\tilde{\tau}}{2p} - \frac{p}{2}\log\left(\frac{\vartheta_4\left(\left.\frac{\mathrm{i}\,\tilde{\tau}}{p}\right| \tilde{q}\right)}{\vartheta_4\left(0|\, \tilde{q}\right)}\right) \tag{6.66}$$

where we have defined $\tau = \mathrm{i}\,K'/K \equiv \pi/\mathrm{i}\,\tilde{\tau}$ and $\tilde{q} = \mathrm{e}^{\pi\,\mathrm{i}\,\tilde{\tau}}$. At leading order in the solution we have

$$\tilde{\tau} = -\frac{t\,p}{2} = -t_s - \frac{2t_s}{p-2}\,. \tag{6.67}$$

The corrections coming from the theta-functions are exponentially suppressed at this level.

To explore the subleading order, it is convenient to write our equation as

$$\tilde{\tau} = -t_s - \frac{2t_s}{p-2} - p^2\log\left(\frac{\vartheta_4\left(\left.\frac{\mathrm{i}\,\tilde{\tau}}{p}\right| \tilde{q}\right)}{\vartheta_4\left(0|\, \tilde{q}\right)}\right) \tag{6.68}$$

and proceed iteratively. The first non-trivial order is obtained by substituting back the zeroth-order solution (6.67) into eq. (6.68) to get

$$\tilde{\tau} = -t_s - \frac{2t_s}{p-2} + 4p^2\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-n\,t_s - \frac{2n\,t_s}{p-2}}}{1 - \mathrm{e}^{-2n\,t_s - \frac{4n\,t_s}{p-2}}}\sinh^2\left(\frac{n\,t_s}{p-2}\right)\,, \tag{6.69}$$

and so on. Here we have used the standard expansion of the logarithm of a ratio of two theta-function as a q-series. It is evident that the solution for the modular parameter $\tilde{\tau}$ and thus the partition function nicely organizes into a double expansion in the two moduli e^{-t_s} and $\mathrm{e}^{-2t_s/(p-2)}$ as expected from string theory.

7. Conclusions

In this paper we have discussed the existence of a large N phase transition in q-deformed Yang-Mills theory on the sphere which occurs for integer parameters $p > 2$ at the critical 't Hooft coupling constant $t_c = p \log(\sec(\frac{\pi}{p})^2)$. The transition appears to be of third order and to separate a weak-coupling phase, where nonperturbative contributions (instantons or better, in the present context, flat connections) are suppressed, from a strong-coupling regime where they are enhanced. This occurs in complete analogy with the undeformed case and we can consider the new phase transition as a q-deformed version of the familiar Douglas-Kazakov transition on the sphere. The two phenomena are in fact smoothly connected. In ordinary two-dimensional Yang-Mills theory, the strong-coupling region is described by the Gross-Taylor string theory while the weak-coupling regime is essentially trivial. Instead, in the q-deformed case both regimes seem to have a string description due to the intimate relation with closed topological string theory.

In the weak-coupling phase, which reduces the gauge theory to the trivial flat connection sector of Chern-Simons theory on the Lens space $L_p = S^3/\mathbb{Z}_p$, we found the familiar resolved conifold geometry that is expected by the well-known geometric transition. On the strong-coupling side, the theory should reproduce the black hole partition function in terms of closed topological string amplitudes on $X = \mathcal{O}(p-2) \oplus \mathcal{O}(-p) \to \mathbb{P}^1$. In particular, we expect the emergence of chiral-antichiral dynamics responsible for the appearance of $|Z_{\rm top}|^2$, along with fiber D-brane contributions and a sum over Ramond-Ramond fluxes. Our strong-coupling solution which is based on a symmetric ansatz probably misses the non-trivial fluxes, because it only computes the $Q = 0$ sector of the U(1) component of the gauge group, but it should capture the correct Calabi-Yau geometry as well as the D-brane contributions. These aspects will be the subjects of a subsequent paper devoted to the comparison of the strong-coupling regime with the topological string amplitudes and the black hole partition function [28]. Here we concentrated our attention on the construction of the strong-coupling solution based on a two-cut ansatz. We proved that the relevant equations always admit a unique solution for $p > 2$, and by constructing the distribution function above the critical point we found a third-order phase transition. The most surprising result is the absence of phase transitions for $p \leq 2$. In these cases the gauge theory always remains in the weak-coupling phase and the strong-coupling equations have no solutions.

It is tempting at this point to speculate about the meaning and possible explanations of these results by resorting to the relation with topological string amplitudes. At the same time it is natural to wonder about their implications for the conjecture formulated in [1]. The first observation is that the presence of the phase transition signals from the strong-coupling side a divergence of the string expansion of the large N partition function. In ordinary two-dimensional Yang-Mills theory on the sphere, the Gross-Taylor series diverges at the critical coupling $\lambda_c = \pi^2$ [27] and the entropy of the branched covering maps is considered to be responsible for the critical behaviour. In analogy, we can raise the question about the string degrees of freedom causing the divergence in the q-deformed case. The obvious suspects are the fiber D-branes, for a number of reasons. First of all, as we will show in [28], their contributions are related to the leading area-polynomial behaviour of the Gross-Taylor series that is exactly the source of the divergence. Furthermore, their appearance is intrinsically tied to the non-compactness of the Calabi-Yau manifold [9] and summing over them could simulate the thermodynamic limit which is necessary to drive the phase transition. Another peculiarity about the presence of the fiber D-branes is that the black hole partition function, as written in [6], appears to depend on them in an intrinsic non-holomorphic way. A different Kähler modulus is associated to their contributions which measures the "distance" of the fiber D-branes from the sphere. While the series for the chiral and antichiral partition functions are separately analytic, the series defining the D-brane contributions to the full black hole partition function depends only on the real part of the relevant Kähler parameter which presumably prevents a suitable analytic continuation.

The existence of the phase transition would establish the validity of the conjecture of [1] only above the critical coupling. It could be that the extension of our computation to complex saddle-points and/or to the sum over all U(1) sectors smooths out the transition. If this is not the case, then we should conclude that the exact black hole partition function is equivalent to the related closed topological string theory only over a critical area parameter defining the geometry of the black hole itself. More dramatically, for $p \leq 2$ the topological string description should never be valid, a fact that is quite surprising. It is therefore important to understand the effective degrees of freedom dominating the black hole physics for large charges in the weak-coupling phase and to understand the phase transition at the gravitational level. A particulary intriguing possibility is that the transition truly describes a topology change in the Calabi-Yau background. This topology change could also fit in with the baby universe splitting picture proposed on the torus in [73]. The nature of the phase transition itself could lead to new insights into the relations between strings and reduced models. An interesting third-order phase transition was discovered some years ago [74] in the apparently unrelated context of a quantum mechanical model obtained as a dimensional reduction of four-dimensional $\mathcal{N} = 1$ supersymmetric Yang-Mills theory to a periodic light-cone time. In spite of the completely different starting point, the equations describing the phase transition there bear an impressive similarity with ours in the strong-coupling phase. We believe that this relation is worthy of further pursuit.

Acknowledgments

We thank F. Bonechi, F. Colomo, L. Cornalba, C. Imbimbo, A. Tanzini, M. Tarlini, A. Torrielli for helpful discussions. L.G. would like to thank K. Konishi for having invited him to present part of these results at the miniworkshop "Dynamics of Supersymmetric Gauge Theories", held in Pisa, 7 June 2005. He also acknowledges N. Dorey and S. Benvenuti for

JHEP01(2006)035

very inspiring discussions there. M.C. and S.P. thank H. Ooguri for interesting conversations on these subjects at the "Spring School on Superstring Theory and Related Topics" at ICTP and at the "Summer School on Strings, Gravity and Cosmology" at the Perimeter Institute. Moreover S.P. thanks the organizers of the "Convegno Informale di Fisica Teorica", held in Cortona, May 2005, for giving her the chance to present preliminary results before publication and M.C. thanks the organizers of the 12^{th} meeting of the "North British Mathematical Physics Seminar", York, June 2005 for the invitation to present some of our results. L.G and D.S. thank Heriot-Watt University for a week of kind hospitality and many Edinburgh's pubs. The work of M.C. and R.J.S. was supported in part by PPARC Grant PPA/G/S/2002/00478 and the EU-RTN Network Grant MRTN -CT-2004-005104.

A. Stieltjes-Wigert and Szegò polynomials

This appendix is devoted to collecting some useful results about the Stieltjes-Wigert and Szegò polynomials.

A.1 Definitions and general properties

Define a deformation of integers, called q-numbers, by

$$[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \,. \tag{A.1}$$

For $q \to 1$, $[n]_q = n + O(q-1)$. This is just one of the possible definitions of q-numbers in the literature, but it is the one most convenient for our purposes. By means of eq. (A.1) we can also generalize all functions over the integers. In particular, we can introduce the q-factorial

$$[n]_q! \equiv \prod_{k=1}^{n} [k]_q = \frac{\prod\limits_{k=1}^{n} (q^{-k/2} - q^{k/2})}{(q^{-1/2} - q^{1/2})^n} = \frac{q^{-\frac{n(n-1)}{4}}}{(1-q)^n} \prod_{k=1}^{n} (1-q^k) \equiv \frac{q^{-\frac{n(n-1)}{4}}}{(1-q)^n} (q)_n \tag{A.2}$$

and the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \equiv \frac{[n]_q!}{[n-k]_q! \, [k]_q!} = \frac{q^{-\frac{n(n-1)}{4}}}{q^{-\frac{k(k-1)}{4}} q^{-\frac{(n-k)(n-k-1)}{4}}} \, \frac{(q)_n}{(q)_k (q)_{n-k}} = q^{\frac{k(k-n)}{2}} \, \frac{(q)_n}{(q)_k \, (q)_{n-k}} \,. \tag{A.3}$$

The Szegò polynomials are then defined by

$$S_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \,. \tag{A.4}$$

For $q \to 1$, this expression reduces to

$$S_n(x) \overset{q \to 1}{=} (1+x)^n + O(q-1) \,. \tag{A.5}$$

For generic values of q, one has the Euler identity

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = \prod_{k=0}^{n-1} \left(1 + q^{k - \frac{n-1}{2}} x\right) \,. \tag{A.6}$$

The Stieltjes-Wigert polynomials are defined by

$$W_n(x) = (-1)^n\, q^{n^2+n/2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-n)}{2}-k^2} \left(-q^{-1/2}\, x\right)^k . \tag{A.7}$$

The overall normalization is chosen so that these polynomials are monic,

$$W_n(x) = x^n + \cdots . \tag{A.8}$$

In our specific case the deformation parameter is given by $q = \mathrm{e}^{-g_s}$, and one can show that these polynomials are orthogonal with respect to the measure

$$\mathrm{d}\mu(x) = \mathrm{e}^{-\frac{\log(x)^2}{2g_s}} \frac{\mathrm{d}x}{2\pi} \tag{A.9}$$

on the positive real half-line.

For this, consider the integral

$$\begin{aligned}
&\int_0^\infty \mathrm{d}\mu(x)\; W_n(x)\, W_k(x) \\
&\quad = (-1)^{n+k}\, q^{n^2+n/2+k^2+k/2} \sum_{j=0}^{n}\sum_{l=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} k \\ l \end{bmatrix}_q q^{\frac{j(j-n)}{2}-j^2+\frac{l(l-k)}{2}-l^2} \left(-q^{-1/2}\right)^{j+l} \\
&\qquad \times \int_0^\infty \frac{\mathrm{d}x}{2\pi}\; \mathrm{e}^{-\frac{\log(x)^2}{2g_s}}\; x^{j+l} \\
&\quad = \sqrt{g_s}\,\frac{(-1)^{n+k}}{\sqrt{2\pi}}\, q^{n^2+k^2+\frac{n+k}{2}} \sum_{j=0}^{n}\sum_{l=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} k \\ l \end{bmatrix}_q (-1)^{j+l}\, q^{\frac{1+j+l+2\,j\,l-k\,l-j\,n}{2}} \\
&\quad = \sqrt{g_s}\,\frac{(-1)^{n+k}}{\sqrt{2\pi}}\, q^{n^2+k^2+\frac{n+k+1}{2}} \sum_{j=0}^{n}(-1)^j \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(1-n)/2} \sum_{l=0}^{k} \begin{bmatrix} k \\ l \end{bmatrix}_q \left(-q^{j+(1-k)/2}\right)^l .
\end{aligned} \tag{A.10}$$

With the help of the Euler identity we can compute the sum over k to get

$$\begin{aligned}
&\int_0^\infty \mathrm{d}\mu(x)\; W_n(x)\, W_k(x) \\
&\quad = \sqrt{g_s}\,\frac{(-1)^{n+k}}{\sqrt{2\pi}}\, q^{n^2+k^2+\frac{n+k+1}{2}} \sum_{j=0}^{n}(-1)^j \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(1-n)/2} \prod_{l=0}^{k-1} \left(1 - q^{l+j-k+1}\right) .
\end{aligned} \tag{A.11}$$

Since the computation is symmetric under the exchange $k \leftrightarrow n$, we can assume $n \le k-1$ without loss of generality. This implies that $k-j-1 \in \{0,1,\ldots,k\}$ for $j = 0,1,\ldots,n$. In other words, for all possible values of j the product $\prod_{l=0}^{k-1}\left(1-q^{l+j-k+1}\right)$ always contains a term that vanishes identically and so the whole product is zero. As a consequence, the sum over j vanishes as well.

The symmetry implies that this property extends to all $k \le n-1$. The only remaining possibility is $k = n$. In that case eq. (A.11) reduces to

$$\int_0^\infty \mathrm{d}\mu(x)\; W_n(x)\, W_k(x)$$

$$= \sqrt{\frac{g_s}{2\pi}}\, q^{2n^2+n+\frac{1}{2}} \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(1-n)/2} \prod_{l=0}^{n-1} \left(1 - q^{l+j-n+1}\right) . \qquad (A.12)$$

Again the product vanishes for $0 \le j \le n-1$, because then $0 \le n-1-j \le n-1$. The only non-zero contribution comes from the $j = n$ term which yields

$$\begin{aligned} \int_0^\infty \mathrm{d}\mu(x)\; W_n(x)\, W_k(x) &= \sqrt{\frac{g_s}{2\pi}}\, q^{2n^2+n+\frac{1}{2}}\, (-1)^n\, q^{n(1-n)/2} \prod_{l=1}^{n} \left(1 - q^l\right) \\ &= \sqrt{\frac{g_s}{2\pi}}\, q^{\frac{7}{4}(n^2+n)+\frac{1}{2}} \prod_{l=1}^{n} \left(q^{l/2} - q^{-l/2}\right) \\ &= \sqrt{\frac{g_s}{2\pi}}\, q^{\frac{7}{4}n(n+1)+\frac{1}{2}} \left(q^{1/2} - q^{-1/2}\right)^n [n]_q! \, . \end{aligned} \qquad (A.13)$$

Summarizing, we have proven the orthogonality relation

$$\int_0^\infty \mathrm{d}\mu(x)\; W_n(x)\, W_k(x) = \sqrt{\frac{g_s}{2\pi}}\, q^{\frac{7}{4}n(n+1)+\frac{1}{2}} \left(q^{1/2} - q^{-1/2}\right)^n [n]_q!\; \delta_{nk} \, . \qquad (A.14)$$

In standard matrix model notation this yields the orthogonal polynomial normalization constants

$$h_n = \sqrt{\frac{g_s}{2\pi}}\, q^{\frac{7}{4}n(n+1)+\frac{1}{2}} \left(q^{1/2} - q^{-1/2}\right)^n [n]_q! \, . \qquad (A.15)$$

These polynomials also satisfy the three-term recurrence relation

$$W_{n+1}(x) = \left(x - q^{n+\frac{1}{2}} \left(q^{n+1} + q^n - 1\right)\right) W_n(x) - q^{3n} \left(q^n - 1\right) W_{n-1}(x) \, . \qquad (A.16)$$

A.2 Some useful integrals and Identities

Let us now compute some useful integrals containing the Stieltjes-Wigert polynomials. Consider first

$$\int_0^\infty \mathrm{d}\mu(x)\; W_n(a\,x) = \sqrt{\frac{g_s}{2\pi}}\, (-1)^n\, q^{n^2+(n+1)/2}\, S_n(-q^{\frac{1-n}{2}}\, a) = \sqrt{\frac{g_s}{2\pi}}\, q^{\frac{(n+1)^2}{2}}\, \hat{S}_n(-q^{\frac{1-n}{2}}\, a)$$

where $\hat{S}_n(-q^{\frac{1-n}{2}}\, a)$ is obtained form $S_n(-q^{\frac{1-n}{2}}\, a)$ by normalizing the coefficient of the monomial of highest degree in a to 1. Next we compute

$$\begin{aligned} &\int_0^\infty \mathrm{d}\mu(x)\; \left(W_n(a\,x)\right)^2 \\ &\quad = \sqrt{\frac{g_s}{2\pi}}\, q^{2n^2+n+1/2} \sum_{j=0}^{n} \left(-a\, q^{(1-n)/2}\right)^j \begin{bmatrix} n \\ j \end{bmatrix}_q \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(-q^{j+(1-n)/2}\, a\right)^l \\ &\quad = \sqrt{\frac{g_s}{2\pi}}\, q^{2n^2+n+1/2} \sum_{j=0}^{n} \left(-a\, q^{(1-n)/2}\right)^j \begin{bmatrix} n \\ j \end{bmatrix}_q S_n\left(-q^{j+(1-n)/2}\, a\right) . \end{aligned} \qquad (A.17)$$

To simplify notation, we introduce the variable $\hat{a} = q^{(1-n)/2}\, a$ so that eq. (A.17) reduces to

$$\int_0^\infty \mathrm{d}\mu(x)\; \left(W_n(a\,x)\right)^2 = \sqrt{\frac{g_s}{2\pi}}\, q^{2n^2+n+1/2} \sum_{j=0}^{n} (-\hat{a})^j \begin{bmatrix} n \\ j \end{bmatrix}_q S_n(-q^j\, \hat{a}) \, . \qquad (A.18)$$

B. Elliptic integrals

B.1 Asymptotic expansions of elliptic integrals

We want to compute the large z asymptotic expansion of an integral of the type

$$I = \int_0^\infty \frac{\mathrm{d}t}{t+z}\, f(t)\,, \tag{B.1}$$

where $f(t)$ decays polynomially for large t. In our specific case the function $f(t)$ is given by

$$f(t) = \frac{1}{\sqrt{(t+a)(t+b)(t+c)(t+d)}} \tag{B.2}$$

and it admits the large t expansion

$$f(t) = \frac{1}{t^2} - \frac{1}{2\,t^3}\,(a+b+c+d) + O\left(\frac{1}{t^4}\right). \tag{B.3}$$

The behaviour of I for large z can be evaluated by means of the following theorem which is proven in [76].

Theorem. *Let $f(t)$ be a locally integrable function on $[0,\infty)$ and $\{A_k\}$ a sequence of complex numbers. Suppose that $f(t)$ has a large t expansion for each $n = 1, 2, \ldots$ of the form*

$$f(t) = \sum_{k=0}^{n-1} \frac{A_k}{t^{k+1}} + f_n(t) \tag{B.4}$$

with $f_n(t) = O(t^{-n-1})$ as $t \to \infty$. Then for every $z, \rho > 0$ and $n = 1, 2, \ldots$ one has

$$\int_0^\infty \mathrm{d}t\, \frac{f(t)}{(t+z)^\rho} = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!\, z^{k+\rho}}\, (\rho)_k\, \big[A_k\big(\log(z) - \gamma - \Psi(k+\rho)\big) + B_k\big] + R_n(z)\,, \tag{B.5}$$

where for $k = 0, 1, 2, \ldots$ the coefficients B_k are given by

$$B_k = A_k \sum_{j=1}^{k} \frac{1}{j} + \lim_{T\to\infty} \left\{ \int_0^T \mathrm{d}t\; t^k\, f(t) - \sum_{j=0}^{k-1} A_j\, \frac{T^{k-j}}{k-j} - A_k\, \log(T) \right\} \equiv A_k \sum_{j=1}^{k} \frac{1}{j} + \mathsf{B}_k \tag{B.6}$$

and empty sums are understood as zero. The remainder term is given by

$$R_n(z) = (\rho)_n \int_0^\infty \mathrm{d}t\, \frac{f_{n,n}(t)}{(t+z)^{n+\rho}} \tag{B.7}$$

where

$$f_{n,n}(t) = \frac{(-1)^n}{(n-1)!} \int_t^\infty \mathrm{d}u\; (u-t)^{n-1}\, f_n(u)\,. \tag{B.8}$$

The constant γ is the Euler constant, Ψ is the digamma function and $(\rho)_k$ is the Pochhammer symbol.

JHEP01(2006)035

This result simplifies drastically when $\rho = 1$. In that case one has

$$(1)_k = k!\,, \qquad \Psi(k+1) = -\gamma + \sum_{j=1}^{k} \frac{1}{j}\,, \tag{B.9}$$

and the asymptotic expansion takes the very simple form

$$\int_0^\infty \mathrm{d}t\, \frac{f(t)}{t+z} = \sum_{k=0}^{n-1} \frac{(-1)^k}{z^{k+1}} \left[A_k \log(z) + \mathsf{B}_k\right] + R_n(z)\,. \tag{B.10}$$

This result can also be expressed in the more convenient form[1]

$$\int_0^\infty \mathrm{d}t\, \frac{f(t)}{t+z} + f(-z)\log(z) = \sum_{k=0}^{n-1} \frac{(-1)^k\, \mathsf{B}_k}{z^{k+1}} + R_n(z)\,. \tag{B.11}$$

In other words, the combinations on the right-hand sides of these equations admit an expansion in powers of $\frac{1}{z}$ alone at $z \to \infty$. This ensures that our resolvent has only physical cuts.

We now apply this result to evaluate the asymptotic series of

$$\begin{aligned}
\mathcal{I} &= \frac{1}{z}\int_{-\infty}^{0} \frac{\mathrm{d}w}{z-w} \sqrt{\frac{(z-\mathrm{e}^{c_-})(z-\mathrm{e}^{c_+})(z-\mathrm{e}^{d_+})(z-\mathrm{e}^{d_-})}{(w-\mathrm{e}^{c_-})(w-\mathrm{e}^{c_+})(w-\mathrm{e}^{d_+})(w-\mathrm{e}^{d_-})}} + \frac{\log(z)}{z} \\
&= \frac{1}{z}\int_{0}^{\infty} \frac{\mathrm{d}w}{z+w} \sqrt{\frac{(z-\mathrm{e}^{c_-})(z-\mathrm{e}^{c_+})(z-\mathrm{e}^{d_+})(z-\mathrm{e}^{d_-})}{(w+\mathrm{e}^{c_-})(w+\mathrm{e}^{c_+})(w+\mathrm{e}^{d_+})(w+\mathrm{e}^{d_-})}} + \frac{\log(z)}{z} \\
&= \frac{\sqrt{(z-\mathrm{e}^{c_-})(z-\mathrm{e}^{c_+})(z-\mathrm{e}^{d_+})(z-\mathrm{e}^{d_-})}}{z} \left(\frac{\mathsf{B}_0}{z} - \frac{\mathsf{B}_1}{z^2} + \cdots\right) \\
&= \mathsf{B}_0 - \frac{1}{z}\left(\mathsf{B}_1 + \frac{\mathsf{B}_0}{2}\left(\mathrm{e}^{c_-} + \mathrm{e}^{c_+} + \mathrm{e}^{d_-} + \mathrm{e}^{d_+}\right)\right) + O\left(\frac{1}{z^2}\right).
\end{aligned} \tag{B.12}$$

Since $A_0 = 0$, $A_1 = 1$ and $A_2 = -\frac{1}{2}(a+b+c+d)$ in this case, the coefficients of the expansion are easily given in terms of elliptic integrals

$$\mathsf{B}_0 = \int_0^\infty \frac{\mathrm{d}t}{\sqrt{(t+\mathrm{e}^{c_-})(t+\mathrm{e}^{c_+})(t+\mathrm{e}^{d_+})(t+\mathrm{e}^{d_-})}}\,, \tag{B.13}$$

$$\begin{aligned}
\mathsf{B}_1 &= \lim_{T\to\infty}\left(\int_0^T \mathrm{d}t\, \frac{t}{\sqrt{(t+\mathrm{e}^{c_-})(t+\mathrm{e}^{c_+})(t+\mathrm{e}^{d_+})(t+\mathrm{e}^{d_-})}} - \log(T)\right) \\
&= -\int_0^\infty \mathrm{d}t\, \log(t)\, \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^2}{\sqrt{(t+\mathrm{e}^{c_-})(t+\mathrm{e}^{c_+})(t+\mathrm{e}^{d_+})(t+\mathrm{e}^{d_-})}}\right) \\
&= \int_0^\infty \mathrm{d}t\, \log(t)\, \frac{\mathrm{d}}{\mathrm{d}t}\big(t^2 f(t)\big)\,.
\end{aligned} \tag{B.14}$$

The other integral function whose asymptotic expansion plays an important role is

$$\mathcal{H} = \int_{\mathrm{e}^{c_-}}^{\mathrm{e}^{c_+}} \frac{\mathrm{d}w}{w(z-w)} \sqrt{\frac{(z-\mathrm{e}^{c_-})(z-\mathrm{e}^{c_+})(z-\mathrm{e}^{d_+})(z-\mathrm{e}^{d_-})}{(w-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-w)(\mathrm{e}^{d_+}-w)(w-\mathrm{e}^{d_-})}}$$

[1] Note that the coefficient of $\log(z)$ is the large z asymptotic expansion of $-f(-z)$.

$$
\begin{aligned}
&= \frac{1}{z} \int_{\mathrm{e}^{c_-}}^{\mathrm{e}^{c_+}} \frac{\mathrm{d}w}{w} \sqrt{\frac{(z-\mathrm{e}^{c_-})(z-\mathrm{e}^{c_+})(z-\mathrm{e}^{d_+})(z-\mathrm{e}^{d_-})}{(w\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-w)(\mathrm{e}^{d_+}-w)(w-\mathrm{e}^{d_-})}} \\
&\quad + \frac{1}{z^2} \int_{\mathrm{e}^{c_-}}^{\mathrm{e}^{c_+}} \mathrm{d}w \sqrt{\frac{(z-\mathrm{e}^{c_-})(z-\mathrm{e}^{c_+})(z-\mathrm{e}^{d_+})(z-\mathrm{e}^{d_-})}{(w-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-w)(\mathrm{e}^{d_+}-w)(w-\mathrm{e}^{d_-})}} \\
&\quad + \frac{1}{z^3} \int_{\mathrm{e}^{c_-}}^{\mathrm{e}^{c_+}} \mathrm{d}w\, w \sqrt{\frac{(z-\mathrm{e}^{c_-})(z-\mathrm{e}^{c_+})(z-\mathrm{e}^{d_+})(z-\mathrm{e}^{d_-})}{(w-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-w)(\mathrm{e}^{d_+}-w)(w-\mathrm{e}^{d_-})}} + \cdots \\
&= \mathcal{H}_{-1}\, z + \mathcal{H}_0 - \frac{1}{2}\left(\mathrm{e}^{c_-} + \mathrm{e}^{c_+} + \mathrm{e}^{d_-} + \mathrm{e}^{d_+}\right) \mathcal{H}_{-1} \\
&\quad + \frac{1}{z}\left[\frac{1}{2}\left(\mathrm{e}^{c_-+c_+} + \mathrm{e}^{c_-+d_-} + \mathrm{e}^{c_++d_-} + \mathrm{e}^{c_-+d_+} + \mathrm{e}^{c_++d_+} + \mathrm{e}^{d_-+d_+}\right.\right. \\
&\quad \left.\left. - \frac{1}{4}\left(\mathrm{e}^{c_-} + \mathrm{e}^{c_+} + \mathrm{e}^{d_-} + \mathrm{e}^{d_+}\right)^2\right) \mathcal{H}_{-1} - \frac{1}{2}\left(\mathrm{e}^{c_-} + \mathrm{e}^{c_+} + \mathrm{e}^{d_-} + \mathrm{e}^{d_+}\right) \mathcal{H}_0 + \mathcal{H}_1\right] \\
&\quad + O\left(\frac{1}{z^2}\right)
\end{aligned} \tag{B.15}
$$

where the constants $\mathcal{H}_n$ are defined by

$$
\mathcal{H}_n = \int_{\mathrm{e}^{c_-}}^{\mathrm{e}^{c_+}} \mathrm{d}w\, \frac{w^n}{\sqrt{(w-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-w)(\mathrm{e}^{d_+}-w)(w-\mathrm{e}^{d_-})}}\,. \tag{B.16}
$$

The constant $\mathcal{H}_1$ is not independent of $\mathcal{H}_{-1}$. Using the change of variable $w \to \mathrm{e}^{2t^2/a}/w$ we find

$$
\mathcal{H}_1 = \mathrm{e}^{2t^2/a}\, \mathcal{H}_{-1}\,. \tag{B.17}
$$

Finally, we consider the integral

$$
\begin{aligned}
\hat{\mathcal{C}} &= \frac{1}{z}\left(\int_{-\infty}^{-\epsilon} \frac{\mathrm{d}w}{w\sqrt{(w-\mathrm{e}^{c_-})(w-\mathrm{e}^{c_+})(w-\mathrm{e}^{d_+})(w-\mathrm{e}^{d_-})}} - \mathrm{e}^{-2\,t^2/a}\,\log(\epsilon)\right) \\
&= -\frac{1}{z}\int_{-\infty}^{0} \mathrm{d}w\, \log|w|\, \frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{1}{\sqrt{(w-\mathrm{e}^{c_-})(w-\mathrm{e}^{c_+})(w-\mathrm{e}^{d_+})(w-\mathrm{e}^{d_-})}}\right) \\
&= \frac{1}{z}\int_{0}^{\infty} \mathrm{d}w\, \log(w)\, \frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{1}{\sqrt{(w+\mathrm{e}^{c_-})(w+\mathrm{e}^{c_+})(w+\mathrm{e}^{d_+})(w+\mathrm{e}^{d_-})}}\right) \\
&= \frac{\mathcal{C}_0}{z} = -\,\mathrm{e}^{-2t^2/a}\, \frac{\left(\frac{2t^2}{a} + \mathsf{B}_1\right)}{z}\,.
\end{aligned} \tag{B.18}
$$

The last equality relating $\mathcal{C}_0$ to B_1 is obtained through the usual change of variable $w \to \mathrm{e}^{2t^2/a}/w$. The asymptotic expansion of $\mathcal{C}$ is then given by

$$
\begin{aligned}
\mathcal{C} &= \sqrt{(z-\mathrm{e}^{c_-})(z-\mathrm{e}^{c_+})(z-\mathrm{e}^{d_-})(z-\mathrm{e}^{d_+})}\;\hat{\mathcal{C}} \\
&= z\,\mathcal{C}_0 - \frac{\mathcal{C}_0}{2}\left(\mathrm{e}^{c_-} + \mathrm{e}^{c_+} + \mathrm{e}^{d_-} + \mathrm{e}^{d_+}\right) + \frac{1}{2\,z}\left(\mathrm{e}^{c_-+c_+} + \mathrm{e}^{c_-+d_-} + \mathrm{e}^{c_++d_-} + \mathrm{e}^{c_-+d_+}\right. \\
&\qquad \left. + \mathrm{e}^{c_++d_+} + \mathrm{e}^{d_-+d_+} - \frac{1}{4}\left(\mathrm{e}^{c_-} + \mathrm{e}^{c_+} + \mathrm{e}^{d_-} + \mathrm{e}^{d_+}\right)^2\right) \mathcal{C}_0 + O\left(\frac{1}{z}\right)^2.
\end{aligned} \tag{B.19}
$$

B.2 Evaluation of elliptic integrals

We want to compute the elliptic integral

$$\int_{\mathrm{e}^{c_-}}^{\mathrm{e}^{c_+}} \frac{\mathrm{d}w}{\sqrt{(w-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-w)(\mathrm{e}^{d_+}-w)(w-\mathrm{e}^{d_-})}}\,. \tag{B.20}$$

Its modulus is given by

$$k=\sqrt{\frac{(\mathrm{e}^{c_+}-\mathrm{e}^{c_-})(\mathrm{e}^{d_+}-\mathrm{e}^{d_-})}{(\mathrm{e}^{d_+}-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-\mathrm{e}^{d_-})}}=\sqrt{\frac{(\mathrm{e}^{a\,c/p}-\mathrm{e}^{-a\,c/p})(\mathrm{e}^{a\,d/p}-\mathrm{e}^{-a\,d/p})}{(\mathrm{e}^{a\,d/p}-\mathrm{e}^{-a\,c/p})(\mathrm{e}^{a\,c/p}-\mathrm{e}^{-a\,d/p})}}\overset{p\to\infty}{=}2\,\frac{\sqrt{c\,d}}{c+d} \tag{B.21}$$

and the result is

$$\int_{\mathrm{e}^{c_-}}^{\mathrm{e}^{c_+}} \frac{\mathrm{d}w}{\sqrt{(w-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-w)(\mathrm{e}^{d_+}-w)(w-\mathrm{e}^{d_-})}}=\frac{2K\,(k)}{\sqrt{(\mathrm{e}^{d_+}-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-\mathrm{e}^{d_-})}}\,. \tag{B.22}$$

With the same definitions, we have

$$\begin{aligned}
&\int_{\mathrm{e}^{c_-}}^{\mathrm{e}^{c_+}} \frac{\mathrm{d}w}{w}\,\frac{1}{\sqrt{(w-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-w)(\mathrm{e}^{d_+}-w)(w-\mathrm{e}^{d_-})}}\\
&\quad=\mathrm{e}^{-2t^2/a}\int_{\mathrm{e}^{c_-}}^{\mathrm{e}^{c_+}}\mathrm{d}w\,\frac{w}{\sqrt{(w-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-w)(\mathrm{e}^{d_+}-w)(w-\mathrm{e}^{d_-})}}\\
&\quad=\frac{2\,\mathrm{e}^{-2t^2/a}}{\sqrt{(\mathrm{e}^{d_+}-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-\mathrm{e}^{d_-})}}\left[(\mathrm{e}^{c_-}-\mathrm{e}^{d_-})\,\Pi\left(\frac{\mathrm{e}^{c_+}-\mathrm{e}^{c_-}}{\mathrm{e}^{c_+}-\mathrm{e}^{d_-}},k\right)+\mathrm{e}^{d_-}K(k)\right].
\end{aligned} \tag{B.23}$$

The next integral to be computed is

$$\int_0^\infty \mathrm{d}w\,\frac{1}{\sqrt{(w+\mathrm{e}^{c_-})(w+\mathrm{e}^{c_+})(w+\mathrm{e}^{d_+})(w+\mathrm{e}^{d_-})}}=\frac{2\big(F(\nu_\infty,k)-F(\nu_0,k)\big)}{\sqrt{(\mathrm{e}^{d_+}-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-\mathrm{e}^{d_-})}} \tag{B.24}$$

with

$$\nu_0=\arcsin\left(\sqrt{\frac{(\mathrm{e}^{d_+}-\mathrm{e}^{c_-})\,\mathrm{e}^{d_-}}{(\mathrm{e}^{d_+}-\mathrm{e}^{d_-})\,\mathrm{e}^{c_-}}}\right)=\arcsin\left(\sqrt{\frac{1}{2}+\frac{c}{2\,d}}\right)-\sqrt{d^2-c^2}\,\left(\frac{t}{4}+\frac{c\,t^2}{48}\right)+O(t^3) \tag{B.25}$$

and

$$\nu_\infty=\arcsin\left(\sqrt{\frac{\mathrm{e}^{d_+}-\mathrm{e}^{c_-}}{\mathrm{e}^{d_+}-\mathrm{e}^{d_-}}}\right)=\arcsin\left(\sqrt{\frac{1}{2}+\frac{c}{2\,d}}\right)+\sqrt{d^2-c^2}\,\left(\frac{t}{4}-\frac{c\,t^2}{48}\right)+O(t^3)\,. \tag{B.26}$$

In the limit $t\to 0$ we thus have

$$F(\nu_\infty,k)-F(\nu_0,k)=\frac{1}{2}\,(c+d)\,t+O(t^2)\,. \tag{B.27}$$

Finally, consider the integral

$$\mathcal{C}_0 = \int_0^\infty \mathrm{d}w \;\log(w)\,\frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{1}{\sqrt{(w+\mathrm{e}^{\,c_-})(w+\mathrm{e}^{\,c_+})(w+\mathrm{e}^{\,d_+})(w+\mathrm{e}^{\,d_-})}}\right). \tag{B.28}$$

Instead of evaluating this, we shall evaluate B_1. $\mathcal{C}_0$ may then be recovered through the identity $\mathcal{C}_0 = -\,\mathrm{e}^{\,-2t^2/a}\,(\mathsf{B}_1 + 2t^2/a)$. We have

$$\begin{aligned}
\mathsf{B}_1 &= -\int_0^\infty \mathrm{d}w \;\log(w)\,\frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{w^2}{\sqrt{(w+\mathrm{e}^{\,c_-})(w+\mathrm{e}^{\,c_+})(w+\mathrm{e}^{\,d_+})(w+\mathrm{e}^{\,d_-})}}\right)\\
&= \lim_{\epsilon\to 0}\left(\log(\epsilon) + \int_0^{1/\epsilon}\mathrm{d}w\;\frac{w}{\sqrt{(w+\mathrm{e}^{\,c_-})(w+\mathrm{e}^{\,c_+})(w+\mathrm{e}^{\,d_+})(w+\mathrm{e}^{\,d_-})}}\right)\\
&= \lim_{\epsilon\to 0}\left[\log(\epsilon) + \frac{2}{\sqrt{(\mathrm{e}^{\,c_+}-\mathrm{e}^{\,d_-})(\mathrm{e}^{\,d_+}-\mathrm{e}^{\,c_-})}}\left\{(\mathrm{e}^{\,c_-}-\mathrm{e}^{\,d_-})\left(\Pi\left(\nu_{1/\epsilon},\frac{\mathrm{e}^{\,d_+}-\mathrm{e}^{\,d_-}}{\mathrm{e}^{\,d_+}-\mathrm{e}^{\,c_-}},k\right)\right.\right.\right.\\
&\quad \left.\left.\left. -\,\Pi\left(\nu_0,\frac{\mathrm{e}^{\,d_+}-\mathrm{e}^{\,d_-}}{\mathrm{e}^{\,d_+}-\mathrm{e}^{\,c_-}},k\right)\right)\right\} - \mathrm{e}^{\,c_-}\left(F(\nu_{1/\epsilon},k) - F(\nu_0,k)\right)\right]
\end{aligned} \tag{B.29}$$

where

$$\nu_{1/\epsilon} = \arcsin\left(\sqrt{\frac{(\mathrm{e}^{\,d_+}-\mathrm{e}^{\,c_-})(1+\epsilon\;\mathrm{e}^{\,d_-})}{(\mathrm{e}^{\,d_+}-\mathrm{e}^{\,d_-})(1+\epsilon\;\mathrm{e}^{\,c_-})}}\right). \tag{B.30}$$

The elliptic integrals of the third kind appearing above are hyperbolic of type II [75], since

$$n = \frac{\mathrm{e}^{\,d_+}-\mathrm{e}^{\,d_-}}{\mathrm{e}^{\,d_+}-\mathrm{e}^{\,c_-}} = \frac{1}{\sin^2(\nu_\infty)} > 1\,. \tag{B.31}$$

We want to reduce them to hyperbolic integrals of type I having $0 < n < k$. This is achieved through the identity

$$\Pi(\nu,n,k) = -\Pi\left(\nu,\frac{k^2}{n},k\right) + F(\nu,k) + \frac{1}{2p_1}\,\log\left(\frac{\Delta(\nu)+p_1\tan(\nu)}{\Delta(\nu)-p_1\tan(\nu)}\right) \tag{B.32}$$

with $\Delta(\nu) = \sqrt{1-k^2\sin^2(\nu)}$ and $p_1 = \sqrt{(n-1)(1-\frac{k^2}{n})}$. Then

$$\begin{aligned}
\Pi(\nu_{1/\epsilon},n,k) - \Pi(\nu_0,n,k) &\overset{\epsilon\to 0}{=} \Pi\left(\nu_0,\frac{k^2}{n},k\right) - \Pi\left(\nu_\infty,\frac{k^2}{n},k\right) + F(\nu_\infty,k) - F(\nu_0,k)\\
&\quad + \frac{1}{2p_1}\log\left(\frac{\Delta(\nu_{1/\epsilon})+p_1\tan(\nu_{1/\epsilon})}{\Delta(\nu_{1/\epsilon})-p_1\tan(\nu_{1/\epsilon})}\,\frac{\Delta(\nu_0)-p_1\tan(\nu_0)}{\Delta(\nu_0)+p_1\tan(\nu_0)}\right).
\end{aligned} \tag{B.33}$$

The argument of the logarithm to first order in ϵ is

$$\frac{\Delta(\nu_{1/\epsilon})+p_1\tan(\nu_{1/\epsilon})}{\Delta(\nu_{1/\epsilon})-p_1\tan(\nu_{1/\epsilon})}\,\frac{\Delta(\nu_0)-p_1\tan(\nu_0)}{\Delta(\nu_0)+p_1\tan(\nu_0)} = \frac{\mathrm{e}^{\,-t(c+d)+\frac{t^2}{a}}}{4}\left(\mathrm{e}^{\,c\,t}+\mathrm{e}^{\,d\,t}\right)\left(1+\mathrm{e}^{\,(c+d)\,t}\right)\epsilon\,. \tag{B.34}$$

Since

$$\frac{2\left(\mathrm{e}^{c_-}-\mathrm{e}^{d_-}\right)}{\sqrt{(\mathrm{e}^{d_+}-\mathrm{e}^{c_-})(\mathrm{e}^{c_+}-\mathrm{e}^{d_-})}}\,\frac{1}{2p_1}=-1 \tag{B.35}$$

the ϵ dependence in the argument of the logarithm cancels with the subtraction appearing in the definition of the integral. We can therefore safely drop it. The original integral is then equal to

$$\begin{aligned}
\mathsf{B}_1 = {} & \frac{2}{\sqrt{(\mathrm{e}^{c_+}-\mathrm{e}^{d_-})(\mathrm{e}^{d_+}-\mathrm{e}^{c_-})}}\left((\mathrm{e}^{c_-}-\mathrm{e}^{d_-})\left\{\Pi\left(\nu_0,\frac{\mathrm{e}^{c_+}-\mathrm{e}^{c_-}}{\mathrm{e}^{c_+}-\mathrm{e}^{d_-}},k\right)\right.\right.\\
& \left.\left.-\,\Pi\left(\nu_\infty,\frac{\mathrm{e}^{c_+}-\mathrm{e}^{c_-}}{\mathrm{e}^{c_+}-\mathrm{e}^{d_-}},k\right)\right\}-\mathrm{e}^{d_-}\left(F(\nu_\infty,k)-F(\nu_0,k)\right)\right)\\
& -\log\left(\frac{\mathrm{e}^{-t(c+d)+t^2/a}}{4}\,(\mathrm{e}^{ct}+\mathrm{e}^{dt})\,(1+\mathrm{e}^{(c+d)\,t})\right)
\end{aligned} \tag{B.36}$$

and thus

$$\begin{aligned}
\mathcal{C}_0 = -\,\mathrm{e}^{-2t^2/a} & \left[\frac{2}{\sqrt{(\mathrm{e}^{c_+}-\mathrm{e}^{d_-})(\mathrm{e}^{d_+}-\mathrm{e}^{c_-})}}\left((\mathrm{e}^{c_-}-\mathrm{e}^{d_-})\left\{\Pi\left(\nu_0,\frac{\mathrm{e}^{c_+}-\mathrm{e}^{c_-}}{\mathrm{e}^{c_+}-\mathrm{e}^{d_-}},k\right)-\right.\right.\right.\\
& \left.\left.-\,\Pi\left(\nu_\infty,\frac{\mathrm{e}^{c_+}-\mathrm{e}^{c_-}}{\mathrm{e}^{c_+}-\mathrm{e}^{d_-}},k\right)\right\}-\mathrm{e}^{d_-}\left(F(\nu_\infty,k)-F(\nu_0,k)\right)\right)\\
& \left.-\log\left(\frac{1}{4}(\mathrm{e}^{d_-}+\mathrm{e}^{c_-})\,(\mathrm{e}^{-d_+-d_-}+\mathrm{e}^{-c_--d_-})\right)\right]\,.
\end{aligned} \tag{B.37}$$

C. Series expansions

C.1 Saddle-point equation

We will now derive eq. (6.52) from eq. (6.50). It will prove useful to introduce the modular parameter $q=\exp\left(-\pi\,K(k')/K(k)\right)$. Let us consider the first term on the r.h.s. of eq. (6.50). By using the expansions

$$\begin{aligned}
\log\left(\mathrm{cn}\left(\frac{2K(k)}{p},\,k\right)\right) &= \log\left(\cos\left(\frac{\pi}{p}\right)\right)-4\sum_{n=1}^{\infty}\frac{1}{n}\,\frac{q^n}{1+(-1)^n\,q^n}\,\sin^2\left(\frac{n\,\pi}{p}\right)\,,\\
\log\left(\mathrm{dn}\left(\frac{2K(k)}{p},\,k\right)\right) &= -8\sum_{n=1}^{\infty}\frac{1}{2n-1}\,\frac{q^{2n-1}}{1-q^{2(2n-1)}}\,\sin^2\left(\frac{(2n-1)\,\pi}{p}\right)\,,
\end{aligned} \tag{C.1}$$

it can be rewritten as

$$\begin{aligned}
\log\left(\frac{\mathrm{dn}\left(\frac{2K(k)}{p},k\right)}{\mathrm{cn}^2\left(\frac{2K(k)}{p},k\right)}\right) = {} & -2\log\left(\cos\left(\frac{\pi}{p}\right)\right)+4\sum_{n=1}^{\infty}\frac{1}{n}\,\frac{q^{2n}}{1+q^{2n}}\,\sin^2\left(\frac{2n\,\pi}{p}\right)\\
& +8\sum_{n=1}^{\infty}\frac{1}{2n-1}\,\frac{q^{2(2n-1)}}{1-q^{2(2n-1)}}\,\sin^2\left(\frac{(2n-1)\,\pi}{p}\right)\,.
\end{aligned} \tag{C.2}$$

To expand the elliptic integral of the third kind in the second term on the r.h.s. of eq. (6.50) requires a bit more of work. We introduce a set of auxiliary parameters given by

$$\epsilon = \arcsin\left(\frac{\sqrt{n}}{k}\right) = \arcsin\left(\mathrm{sn}\left(\frac{\hat{x}}{2},k\right)\right), \tag{C.3}$$

$$\beta = \frac{\pi}{2}\,\frac{F(\epsilon,k)}{K(k)} = \frac{\pi}{2}\,\frac{F\left[\arcsin\left(\mathrm{sn}\left(\frac{\hat{x}}{2},k\right)\right),k\right]}{K(k)} = \frac{\pi}{2K(k)}\,\frac{\hat{x}}{2} = \frac{\pi}{2}\,\frac{p+2}{2p}, \tag{C.4}$$

$$v = \frac{\pi}{2}\,\frac{F(\nu,k)}{K(k)}, \tag{C.5}$$

$$\delta_1 = \sqrt{\frac{n}{(1-n)(k^2-n)}} = \frac{\mathrm{sn}\left(\frac{\hat{x}}{2},k\right)}{\mathrm{dn}\left(\frac{\hat{x}}{2},k\right)\,\mathrm{cn}\left(\frac{\hat{x}}{2},k\right)}. \tag{C.6}$$

A generic elliptic integral of the third kind with $0 \le n \le k^2$ can be represented as

$$\Pi(\nu,n,k) = \delta_1\left[-\frac{1}{2}\,\log\left(\frac{\vartheta_4(v+\beta)}{\vartheta_4(v-\beta)}\right) + v\,\frac{\vartheta_1'(\beta)}{\vartheta_1(\beta)}\right]. \tag{C.7}$$

In the following we will need the expansions in powers of q of the two terms in the r.h.s. of eq. (C.7)

$$\begin{aligned}\frac{1}{2}\,\log\left(\frac{\vartheta_4(v+\beta)}{\vartheta_4(v-\beta)}\right) &= 2\sum_{n=1}^{\infty}\frac{q^n}{n\,(1-q^{2n})}\,\sin(2\,n\,v)\,\sin(2\,n\,\beta),\\ \frac{\vartheta_1'(\beta)}{\vartheta_1(\beta)} &= \cot(\beta) + 4\sin(2\,\beta)\sum_{n=1}^{\infty}\frac{q^{2n}}{1-2\,\cos(2\,\beta)\,q^{2n}+q^{2n}}.\end{aligned} \tag{C.8}$$

By using these expansions we can simplify the relevant combination of elliptic integrals in eq. (6.50) as

$$\begin{aligned}&\Pi(\nu_\infty,\ n\ ,k) - \Pi(\nu_0,n,k) - \frac{2}{p}\Pi(n,k)\\ &= \delta_1\left[-2\sum_{n=1}^{\infty}\frac{q^n}{n\,(1-q^{2n})}\left[\sin(2\,n\,v_\infty) - \sin(2\,n\,v_0) - \frac{2}{p}\sin\pi\,n\right]\sin(2\,n\,\beta)\right.\\ &\quad\left.+\left(v_\infty - v_0 - \frac{\pi}{p}\right)\left(\cot(\beta) + 4\sin(2\,\beta)\sum_{n=1}^{\infty}\frac{q^{2n}}{1-2\cos(2\,\beta)q^{2n}+q^{2n}}\right)\right]\\ &= -4\,\delta_1\sum_{n=1}^{\infty}\frac{q^n}{n\,(1-q^{2n})}\sin\left(\frac{\pi\,n}{p}\right)\cos\left(\frac{\pi\,n}{2}\right)\sin\left(n\,\pi\,\frac{p+2}{2\,p}\right) =\\ &= -2\,\delta_1\sum_{n=1}^{\infty}\frac{q^{2n}}{n\,(1-q^{4n})}\sin^2\left(\frac{2\,\pi\,n}{p}\right),\end{aligned} \tag{C.9}$$

where the second equality holds due to the identities

$$\begin{aligned}v_\infty - v_0 &= \frac{\pi}{2K(k)}(F(\nu_\infty,k) - F(\nu_0,k)) = \frac{\pi}{p},\\ v_\infty + v_0 &= \frac{\pi}{2K(k)}(F(\nu_\infty,k) + F(\nu_0,k)) = \frac{\pi}{2}.\end{aligned} \tag{C.10}$$

and the last equality is a consequence of the fact that only even n contributes.

Finally δ_1 in eq. (C.9) will appear in eq. (6.50) only in the combination

$$\delta_1 \frac{k' \mathrm{cn}(x,k)}{1+\mathrm{sn}(x,k)} = \frac{k'\mathrm{cn}(x,k)}{1+\mathrm{sn}(x,k)} \frac{\mathrm{sn}\left(\frac{\hat{x}}{2},k\right)}{\mathrm{dn}\left(\frac{\hat{x}}{2},k\right)\mathrm{cn}\left(\frac{\hat{x}}{2},k\right)} = \frac{k'\left(\frac{1}{\mathrm{cn}(x,k)} - \frac{\mathrm{sn}(x,k)}{\mathrm{cn}(x,k)}\right)}{\frac{\mathrm{dn}\left(\frac{\hat{x}}{2},k\right)\mathrm{cn}\left(\frac{\hat{x}}{2},k\right)}{\mathrm{sn}\left(\frac{\hat{x}}{2},k\right)}}\,. \tag{C.11}$$

To expand in powers of q we expand separately the numerator and the denominator of the last expression in eq. (C.11):

$$\begin{aligned}
\frac{\mathrm{dn}\left(\frac{\hat{x}}{2},k\right)\mathrm{cn}\left(\frac{\hat{x}}{2},k\right)}{\mathrm{sn}\left(\frac{\hat{x}}{2},k\right)} &= \frac{\pi}{2K(k^2)}\left[\cot\frac{\pi(p+2)}{4p} - 4\sum_{n=1}^{\infty}\frac{q^n}{1+q^n}\sin\frac{n\pi(p+2)}{2p}\right], \\
k'\left(\frac{1}{\mathrm{cn}(x,k)} - \frac{\mathrm{sn}(x,k)}{\mathrm{cn}(x,k)}\right) &= \frac{\pi}{2K(k^2)}\left[\frac{1}{\cos\frac{\pi}{p}} - \tan\frac{\pi}{p} - 4\sum_{n=1}^{\infty}\frac{q^n}{1+q^n}\sin\frac{n\pi(p+2)}{2p}\right] \\
&= \cot\frac{\pi(p+2)}{4p} - 4\sum_{n=1}^{\infty}\frac{q^n}{1+q^n}\sin\frac{n\pi(p+2)}{2p}\,.
\end{aligned} \tag{C.12}$$

Collecting all together, we have

$$\frac{k'\left(\frac{1}{\mathrm{cn}(x,k)} - \frac{\mathrm{sn}(x,k)}{\mathrm{cn}(x,k)}\right)}{\frac{\mathrm{dn}\left(\frac{\hat{x}}{2},k\right)\mathrm{cn}\left(\frac{\hat{x}}{2},k\right)}{\mathrm{sn}\left(\frac{\hat{x}}{2},k\right)}} = 1\,. \tag{C.13}$$

Finally we conclude that the second saddle point equation (6.50) admits the following q-expansion

$$\begin{aligned}
\frac{t}{4} &= \frac{p}{4}\log\left(\frac{\mathrm{dn}(x,k)}{\mathrm{cn}^2(x,k)}\right) + \frac{p}{2}\frac{k'\mathrm{cn}(x,k)}{1+\mathrm{sn}(x,k)}\left[\Pi\left(\nu_\infty,n,k\right) - \Pi\left(\nu_0,n,k\right) - \frac{2}{p}\Pi\left(n,k\right)\right] \\
&= \frac{t_c}{4} + p\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^{2n}}{1+q^{2n}}\sin^2\left(\frac{2\,\pi\,n}{p}\right) + 2p\sum_{n=1}^{\infty}\frac{1}{2n-1}\frac{q^{2(2n-1)}}{1-q^{2(2n-1)}}\sin^2\left(\frac{(2n-1)\,\pi}{p}\right) \\
&\quad -p\sum_{n=1}^{\infty}\frac{q^{2n}}{n(1-q^{4n})}\sin^2\left(\frac{2\,\pi\,n}{p}\right) \\
&= \frac{t_c}{4} - 2p\sum_{n=1}^{\infty}\frac{1}{2\,n}\frac{q^{4n}}{1-q^{4n}}\sin^2\left(\frac{2\,\pi\,n}{p}\right) + 2p\sum_{n=1}^{\infty}\frac{1}{2n-1}\frac{q^{2(2n-1)}}{1-q^{2(2n-1)}}\sin^2\left(\frac{(2n-1)\,\pi}{p}\right) \\
&= \frac{t_c}{4} - 2p\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{q^{2n}}{1-q^{2n}}\sin^2\left(\frac{\pi\,n}{p}\right),
\end{aligned}$$

that leads us to eq. (6.52).

C.2 Endpoints

We want to provide an expansion of the endpoints c and d in terms of the modular parameter q. This can be obtained by considering the system of equations given by eq. (6.46)

$$c+d = \frac{2}{t}\mathrm{arctanh}\left(\mathrm{sn}\left(\frac{2tK(k)}{a},k\right)\right)$$

$$= \frac{2}{t}\text{arctanh}\left(\frac{2\pi}{k\,K(k)}\sum_{n=1}^{\infty}\frac{q^{n-1/2}}{1-q^{2n-1}}\sin\left(\frac{(2n-1)\,\pi}{p}\right)\right), \tag{C.14}$$

and by

$$\begin{aligned} c-d &= \frac{2}{t}\text{arctanh}\left(\text{cn}\left(\frac{p+2}{p}K(k),k\right)\right) \\ &= \frac{2}{t}\text{arctanh}\left(\frac{2\pi}{kK(k)}\sum_{n=1}^{\infty}\frac{(-1)^n q^{n-1/2}}{1+q^{2n-1}}\sin\left(\frac{(2n-1)\pi}{p}\right)\right). \end{aligned} \tag{C.15}$$

Solving for the endpoints we obtain

$$\begin{aligned} d =& \frac{1}{t}\,\text{arctanh}\left(\frac{2\pi}{kK(k)}\sum_{n=1}^{\infty}\frac{q^{n-1/2}}{1-q^{2n-1}}\sin\left(\frac{(2n-1)\pi}{p}\right)\right) \\ &-\frac{1}{t}\,\text{arctanh}\left(\frac{2\pi}{kK(k)}\sum_{n=1}^{\infty}\frac{(-1)^n q^{n-1/2}}{1+q^{2n-1}}\sin\left(\frac{(2n-1)\pi}{p}\right)\right), \end{aligned} \tag{C.16}$$

$$\begin{aligned} c =& \frac{1}{t}\,\text{arctanh}\left(\frac{2\pi}{kK(k)}\sum_{n=1}^{\infty}\frac{q^{n-1/2}}{1-q^{2n-1}}\sin\left(\frac{(2n-1)\pi}{p}\right)\right) \\ &+\frac{1}{t}\,\text{arctanh}\left(\frac{2\pi}{kK(k)}\sum_{n=1}^{\infty}\frac{(-1)^n q^{n-1/2}}{1+q^{2n-1}}\sin\left(\frac{(2n-1)\pi}{p}\right)\right). \end{aligned} \tag{C.17}$$

These are expansions in powers of q since

$$\frac{k\,K(k)}{2\pi} = \left(\sum_{n=1}^{\infty} q^{\left(\frac{2n-1}{2}\right)^2}\right)^2. \tag{C.18}$$

C.3 Distribution function

The goal of this appendix is to provide an expansion in terms of the modular parameter q for the distribution function, that is to derive eq. (6.62) from eq. (6.61).

Let us start with the difference between the two integrals

$$\Pi\left(\nu_\infty, \frac{(\,\mathrm{e}^{\,d_+}-\,\mathrm{e}^{\,d_-})(z-\,\mathrm{e}^{\,c_-})}{(\,\mathrm{e}^{\,d_+}-\,\mathrm{e}^{\,c_-})(z-\,\mathrm{e}^{\,d_-})}, k\right) - \Pi\left(\nu_0, \frac{(\,\mathrm{e}^{\,d_+}-\,\mathrm{e}^{\,d_-})(z-\,\mathrm{e}^{\,c_-})}{(\,\mathrm{e}^{\,d_+}-\,\mathrm{e}^{\,c_-})(z-\,\mathrm{e}^{\,d_-})}, k\right), \tag{C.19}$$

where $z \in [\,\mathrm{e}^{\,c_+},\,\mathrm{e}^{\,d_+}]$. For these values of z they are elliptic integrals of circular type since the parameter

$$n(z) = \frac{(\,\mathrm{e}^{\,d_+}-\,\mathrm{e}^{\,d_-})(z-\,\mathrm{e}^{\,c_-})}{(\,\mathrm{e}^{\,d_+}-\,\mathrm{e}^{\,c_-})(z-\,\mathrm{e}^{\,d_-})} \tag{C.20}$$

always belongs to the region $k^2 \leq n \leq 1$.

They admit the following q-expansion [75]:

$$\Pi(\nu,n,k) = \delta_2\left(\lambda(\beta)-\mu(\beta)\,v\right), \tag{C.21}$$

where

$$\lambda(\beta) = \arctan\left(\tanh\beta\tan v\right) - 2\sum_{n=1}^{\infty}\frac{(-1)^s}{s}\frac{q^{2s}}{1-q^{2s}}\sin(2\,s\,v)\sinh(2\,s\,\beta),$$

$$\mu(\beta) = \frac{\theta_3'(\mathrm{i}\,\beta, q)}{4\theta_3(\mathrm{i}\,\beta, q)} = \frac{\displaystyle\sum_{s=1}^{\infty} s\, q^{s^2} \sinh(2\,s\,\beta)}{\displaystyle\sum_{s=-\infty}^{\infty} q^{s^2} \cosh(2\,s\,\beta)} . \tag{C.22}$$

We have introduced the following set of parameters

$$\begin{aligned}
\epsilon &= \arcsin\left(\sqrt{\frac{1-n(z)}{1-k^2}}\right) , \\
\beta &= \frac{\pi}{2}\frac{F(\epsilon, k')}{K(k)} , \\
v &= \frac{\pi}{2}\frac{F(\nu, k)}{K(k)} , \\
\delta_2 &= \sqrt{\frac{n(z)}{(1-n(z))(n-k^2)}} .
\end{aligned} \tag{C.23}$$

From the explicit form

$$\delta_2 = \frac{\sqrt{(\mathrm{e}^{c_+} - \mathrm{e}^{d_-})(\mathrm{e}^{d_+} - \mathrm{e}^{c_-})}}{(\mathrm{e}^{c_-} - \mathrm{e}^{d_-})}\sqrt{\frac{(z-\mathrm{e}^{c_-})\,(z-\mathrm{e}^{d_-})}{(\mathrm{e}^{d_+}-z)\,(z-\mathrm{e}^{c_+})}} , \tag{C.24}$$

we see that the inverse of δ_2 appears as a common prefactor in the distribution function (6.61). Moreover in our case

$$\begin{aligned}
\beta &= \frac{\pi}{2\mathrm{K}(k)} F\left(\arcsin\left(\sqrt{\frac{(\mathrm{e}^{c_+} - \mathrm{e}^{d_-})\,(\mathrm{e}^{d_+} - z)}{(\mathrm{e}^{d_+} - \mathrm{e}^{c_+})\,(z - \mathrm{e}^{d_-})}}\right), k'\right) \\
&= \frac{\pi}{2\mathrm{K}(k)} \mathrm{sn}^{-1}\left(\sqrt{\frac{(\mathrm{e}^{c_+} - \mathrm{e}^{d_-})\,(\mathrm{e}^{d_+} - z)}{(\mathrm{e}^{d_+} - \mathrm{e}^{c_+})\,(z - \mathrm{e}^{d_-})}}, k'\right) \equiv \beta_1 .
\end{aligned} \tag{C.25}$$

Collecting all these ingredients we can rewrite the eq. (C.19) as

$$\begin{aligned}
&\Pi\left(\nu_\infty, \frac{(\mathrm{e}^{d_+} - \mathrm{e}^{d_-})(z - \mathrm{e}^{c_-})}{(\mathrm{e}^{d_+} - \mathrm{e}^{c_-})(z - \mathrm{e}^{d_-})}, k\right) - \Pi\left(\nu_0, \frac{(\mathrm{e}^{d_+} - \mathrm{e}^{d_-})(z - \mathrm{e}^{c_-})}{(\mathrm{e}^{d_+} - \mathrm{e}^{c_-})(z - \mathrm{e}^{d_-})}, k\right) \\
&= \delta_2 \Bigg(\arctan\left(\tanh\beta_1 \tan v_\infty\right) - \arctan\left(\tanh\beta_1 \tan v_0\right) - 4\,\mu\,(v_\infty - v_0) \\
&\quad - 2\sum_{n=1}^{\infty} \frac{(-1)^s}{s}\frac{q^{2s}}{1-q^{2s}}\big(\sin(2\,s\,v_\infty) - \sin(2\,s\,v_0)\big)\,\sinh(2\,s\,\beta_1)\Bigg) \\
&= \delta_2 \left(\arctan\left(\tan\frac{\pi}{p}\tanh(2\,\beta_1)\right) - 4\frac{\pi}{p}\,\mu(\beta_1) - 2\sum_{s=1}^{\infty}\frac{1}{s}\frac{q^{4s}}{1-q^{4s}}\sin\left(\frac{2\,\pi\,s}{p}\right)\sinh(2\,s\,\beta_1)\right) ,
\end{aligned} \tag{C.26}$$

where we have used eqs. (C.10).

The last term that we have to expand is the complete elliptic integral $\Pi(n,k)$

$$\Pi\left(\frac{(\mathrm{e}^{c_+} - \mathrm{e}^{c_-})(z - \mathrm{e}^{d_-})}{(\mathrm{e}^{c_+} - \mathrm{e}^{d_-})(z - \mathrm{e}^{c_-})}, k\right) - K(k) = \hat{\delta}_2\,\frac{\pi}{2}\,\left(1 - \Lambda(\eta, k)\right) , \tag{C.27}$$

where

$$\eta = \arcsin\left(\sqrt{\frac{1-\hat{n}(z)}{1-k^2}}\right) = \arcsin\left(\sqrt{\frac{(\mathrm{e}^{\,d_+} - \mathrm{e}^{\,c_-})(z - \mathrm{e}^{\,c_+})}{(\mathrm{e}^{\,d_+} - \mathrm{e}^{\,c_+})(z - \mathrm{e}^{\,c_-})}}\right), \tag{C.28}$$

with $\hat{n}(z) = \frac{(\mathrm{e}^{\,c_+} - \mathrm{e}^{\,c_-})(z - \mathrm{e}^{\,d_-})}{(\mathrm{e}^{\,c_+} - \mathrm{e}^{\,d_-})(z - \mathrm{e}^{\,c_-})}$. Since $\hat{n}(z) = k^2/n(z)$, we immediately see that

$$\hat{\delta}_2 = \sqrt{\frac{\hat{n}(z)}{(1-\hat{n}(z))(\hat{n}(z)-k^2)}} = \delta_2\,. \tag{C.29}$$

The function $\Lambda(\eta, k)$ appearing in the expansion of Π is the Heuman lambda-function defined by [75]

$$\Lambda(\eta, k) = \frac{2}{\pi}\left(E(\eta, k')\,K(k) - (K(k) - E(k))\,F(\eta, k')\right)\,. \tag{C.30}$$

The distribution function ρ can be finally written in the form of eq. (6.62)

$$\begin{aligned}\rho(z) = \frac{2}{t\,\pi\,z}\Bigg(&\frac{\pi}{2} - \frac{\pi}{2}\Lambda(\eta, k) + \frac{p}{2}\arctan\left(\tan\frac{\pi}{p}\tanh(2\,\beta_1)\right) - 2\,\pi\,\mu(\beta_1)\\ &-\sum_{s=1}^{\infty}\frac{p}{s}\frac{q^{4s}}{1-q^{4s}}\sin\left(\frac{2\pi s}{p}\right)\sinh(2s\beta_1)\Bigg)\,.\end{aligned} \tag{C.31}$$

This representation obviously holds for $z \in [\,\mathrm{e}^{\,c_+},\, \mathrm{e}^{\,d_+}]$. We are interested in the behavior at the transition point t_c, where $q = k = c = 0$. The critical values of the parameters that enter in the distribution function are

$$\begin{aligned}\beta_1^c &= F\left(\arcsin\left(\mathrm{e}^{\,-d_c t_c/2}\sqrt{\frac{(\mathrm{e}^{\,d_{c+}} - z)}{(z - \mathrm{e}^{\,d_{c-}})}}\right), 1\right)\\ &= \operatorname{arctanh}\left(\mathrm{e}^{\,-d_c t_c/2}\sqrt{\frac{(\mathrm{e}^{\,d_{c+}} - z)}{(z - \mathrm{e}^{\,d_{c-}})}}\right)\\ \eta^c &= \frac{\pi}{2}\,.\end{aligned} \tag{C.32}$$

The second identity implies that $\Lambda(\eta^c, k) = 1$. From these results it is possible to derive the distribution function at the critical point, eq. (6.64).

References

[1] H. Ooguri, A. Strominger and C. Vafa, *Black hole attractors and the topological string*, *Phys. Rev.* **D 70** (2004) 106007 [`hep-th/0405146`].

[2] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Corrections to macroscopic supersymmetric black-hole entropy*, *Phys. Lett.* **B 451** (1999) 309 [`hep-th/9812082`].

[3] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Deviations from the area law for supersymmetric black holes*, *Fortschr. Phys.* **48** (2000) 49 [`hep-th/9904005`].

[4] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Macroscopic entropy formulae and non-holomorphic corrections for supersymmetric black holes*, *Nucl. Phys.* **B 567** (2000) 87 [hep-th/9906094].

[5] T. Mohaupt, *Black hole entropy, special geometry and strings*, *Fortschr. Phys.* **49** (2001) 3 [hep-th/0007195].

[6] M. Aganagic, H. Ooguri, N. Saulina and C. Vafa, *Black holes, q-deformed 2d Yang-Mills and non-perturbative topological strings*, *Nucl. Phys.* **B 715** (2005) 304 [hep-th/0411280].

[7] C. Vafa, *Two dimensional Yang-Mills, black holes and topological strings*, hep-th/0406058.

[8] J. Bryan and R. Pandharipande, *The local Gromov-Witten theory of curves*, math.AG/0411037.

[9] M. Aganagic, A. Neitzke and C. Vafa, *BPS microstates and the open topological string wave function*, hep-th/0504054.

[10] A. Dabholkar, *Exact counting of black hole microstates*, *Phys. Rev. Lett.* **94** (2005) 241301 [hep-th/0409148].

[11] E.P. Verlinde, *Attractors and the holomorphic anomaly*, hep-th/0412139.

[12] G.L. Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Asymptotic degeneracy of dyonic $N = 4$ string states and black hole entropy*, *JHEP* **12** (2004) 075 [hep-th/0412287].

[13] A. Sen, *Black holes, elementary strings and holomorphic anomaly*, *JHEP* **07** (2005) 063 [hep-th/0502126].

[14] A. Sen, *Black holes and the spectrum of half-BPS states in $N = 4$ supersymmetric string theory*, hep-th/0504005.

[15] A. Sen, *Stretching the horizon of a higher dimensional small black hole*, *JHEP* **07** (2005) 073 [hep-th/0505122].

[16] A. Sen, *Black hole entropy function and the attractor mechanism in higher derivative gravity*, *JHEP* **09** (2005) 038 [hep-th/0506177].

[17] A. Dabholkar, F. Denef, G.W. Moore and B. Pioline, *Exact and asymptotic degeneracies of small black holes*, *JHEP* **08** (2005) 021 [hep-th/0502157].

[18] B. Pioline, *BPS black hole degeneracies and minimal automorphic representations*, *JHEP* **08** (2005) 071 [hep-th/0506228].

[19] S. Parvizi and A. Tavanfar, *Minimal redefinition of the OSV ensemble*, hep-th/0508231.

[20] A. Dabholkar, F. Denef, G.W. Moore and B. Pioline, *Precision counting of small black holes*, *JHEP* **10** (2005) 096 [hep-th/0507014].

[21] D.J. Gross, *Two-dimensional QCD as a string theory*, *Nucl. Phys.* **B 400** (1993) 161 [hep-th/9212149].

[22] D.J. Gross and I.W. Taylor, *Twists and wilson loops in the string theory of two-dimensional QCD*, *Nucl. Phys.* **B 403** (1993) 395 [hep-th/9303046].

[23] D.J. Gross and I.W. Taylor, *Two-dimensional QCD is a string theory*, *Nucl. Phys.* **B 400** (1993) 181 [hep-th/9301068].

[24] M.R. Douglas and V.A. Kazakov, *Large-N phase transition in continuum QCD in two-dimensions*, *Phys. Lett.* **B 319** (1993) 219 [hep-th/9305047].

[25] D.J. Gross and A. Matytsin, *Instanton induced large-N phase transitions in two-dimensional and four-dimensional QCD*, *Nucl. Phys.* **B 429** (1994) 50 [hep-th/9404004].

[26] W. Taylor, *Counting strings and phase transitions in 2-d QCD*, hep-th/9404175.

[27] M.J. Crescimanno and W. Taylor, *Large-N phases of chiral QCD in two-dimensions*, *Nucl. Phys.* **B 437** (1995) 3 [hep-th/9408115].

[28] N. Caporaso et al., *Topological strings and large-N phase transitions, II. Chiral expansion of q-deformed Yang-Mills theory*, hep-th/0511043.

[29] C. Beasley and E. Witten, *Non-abelian localization for Chern-Simons theory*, hep-th/0503126.

[30] A. Neitzke and C. Vafa, *Topological strings and their physical applications*, hep-th/0410178.

[31] M. Vonk, *A mini-course on topological strings*, hep-th/0504147.

[32] S. Ferrara, R. Kallosh and A. Strominger, *$N = 2$ extremal black holes*, *Phys. Rev.* **D 52** (1995) 5412 [hep-th/9508072].

[33] A. Strominger, *Macroscopic entropy of $n = 2$ extremal black holes*, *Phys. Lett.* **B 383** (1996) 39 [hep-th/9602111].

[34] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, *Topological amplitudes in string theory*, *Nucl. Phys.* **B 413** (1994) 162 [hep-th/9307158].

[35] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311 [hep-th/9309140].

[36] M. Bershadsky, C. Vafa and V. Sadov, *D-branes and topological field theories*, *Nucl. Phys.* **B 463** (1996) 420 [hep-th/9511222].

[37] C. Vafa and E. Witten, *A strong coupling test of S duality*, *Nucl. Phys.* **B 431** (1994) 3 [hep-th/9408074].

[38] U. Bruzzo, F. Fucito, J.F. Morales and A. Tanzini, *Multi-instanton calculus and equivariant cohomology*, *JHEP* **05** (2003) 054 [hep-th/0211108].

[39] F. Fucito, J.F. Morales and R. Poghossian, *Multi instanton calculus on ale spaces*, *Nucl. Phys.* **B 703** (2004) 518 [hep-th/0406243].

[40] H. Nakajima, *Instantons on ALE spaces, quiver varieties and Kac-Moody algebras*, *Duke Math. J.* **76** (1994) 365.

[41] C. Klimčík, *The formulae of Kontsevich and Verlinde from the perspective of the Drinfeld double*, *Commun. Math. Phys.* **217** (2001) 203 [hep-th/9911239].

[42] M. Blau and G. Thompson, *Derivation of the Verlinde formula from Chern-Simons theory and the G/G model*, *Nucl. Phys.* **B 408** (1993) 345 [hep-th/9305010].

[43] E. Witten, *Two-dimensional gauge theories revisited*, *J. Geom. Phys.* **9** (1992) 303 [hep-th/9204083].

[44] A.A. Migdal, *Recursion equations in gauge field theories*, *Sov. Phys. JETP* **42** (1975) 413 [*Zh. Eksp. Teor. Fiz.* **69** (1975) 810].

[45] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, *The topological vertex*, *Commun. Math. Phys.* **254** (2005) 425 [hep-th/0305132].

[46] D.J. Gross and E. Witten, *Possible third order phase transition in the large N lattice gauge theory*, *Phys. Rev.* **D 21** (1980) 446.

[47] S.R. Wadia, $N = \infty$ *phase transition in a class of exactly soluble model lattice gauge theories*, *Phys. Lett.* **B 93** (1980) 403.

[48] E. Witten, *On quantum gauge theories in two-dimensions*, *Commun. Math. Phys.* **141** (1991) 153.

[49] S. Cordes, G.W. Moore and S. Ramgoolam, *Large-N 2-d Yang-Mills theory and topological string theory*, *Commun. Math. Phys.* **185** (1997) 543 [hep-th/9402107].

[50] M. Tierz, *Soft matrix models and Chern-Simons partition functions*, *Mod. Phys. Lett.* **A 19** (2004) 1365 [hep-th/0212128].

[51] S. de Haro and M. Tierz, *Brownian motion, Chern-Simons theory and 2d Yang-Mills*, *Phys. Lett.* **B 601** (2004) 201 [hep-th/0406093].

[52] S. de Haro and M. Tierz, *Discrete and oscillatory matrix models in Chern-Simons theory*, hep-th/0501123.

[53] G. Szegò, *Orthogonal polynomials*, Colloquium Publications, vol. **23**, Americam Mathematical Society, Providence 1991.

[54] J.A. Minahan and A.P. Polychronakos, *Classical solutions for two-dimensional QCD on the sphere*, *Nucl. Phys.* **B 422** (1994) 172 [hep-th/9309119].

[55] M.R. Douglas, K. Li and M. Staudacher, *Generalized two-dimensional QCD*, *Nucl. Phys.* **B 420** (1994) 118 [hep-th/9401062].

[56] A. Okounkov, N. Reshetikhin and C. Vafa, *Quantum Calabi-Yau and classical crystals*, hep-th/0309208.

[57] T. Okuda, *Derivation of Calabi-Yau crystals from Chern-Simons gauge theory*, *JHEP* **03** (2005) 047 [hep-th/0409270].

[58] M. Mariño, *Chern-Simons theory, matrix integrals and perturbative three-manifold invariants*, *Commun. Math. Phys.* **253** (2004) 25 [hep-th/0207096].

[59] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, *Matrix model as a mirror of Chern-Simons theory*, *JHEP* **02** (2004) 010 [hep-th/0211098].

[60] S. de Haro, *Chern-Simons theory in lens spaces from 2d Yang-Mills on the cylinder*, *JHEP* **08** (2004) 041 [hep-th/0407139].

[61] S. de Haro, *Chern-Simons theory, 2d Yang-Mills and Lie algebra wanderers*, *Nucl. Phys.* **B 730** (2005) 312 [hep-th/0412110].

[62] C. LeBrun, *Counter-examples to the generalized positive action conjecture*, *Commun. Math. Phys.* **118** (1988) 591.

[63] L. Rozansky, *A large K asymptotics of Witten's invariant of Seifert manifolds*, *Commun. Math. Phys.* **171** (1995) 279 [hep-th/9303099].

[64] L. Rozansky, *Residue formulas for the large K asymptotics of Witten's invariants of seifert manifolds: the case of* SU(2), *Commun. Math. Phys.* **178** (1996) 27 [hep-th/9412075].

[65] M. Furuta and B. Steer, *Seifert fibred homology 3-spheres and the Yang-Mills equations on Riemann surfaces with marked points*, *Adv. in Math.* **96** (1992) 38.

JHEP01(2006)035

[66] N. Halmagyi and V. Yasnov, *The spectral curve of the Lens space matrix model*, hep-th/0311117.

[67] N. Halmagyi, T. Okuda and V. Yasnov, *Large-N duality, Lens spaces and the Chern-Simons matrix model*, *JHEP* **04** (2004) 014 [hep-th/0312145].

[68] M. Mariño, *Les Houches lectures on matrix models and topological strings*, hep-th/0410165.

[69] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1415 [hep-th/9811131].

[70] This summation formula can be found at http://functions.wolfram.com/08.05.16.0008.01.

[71] A. Aghamohammadi, M. Alimohammadi and M. Khorrami, *Uniqueness of the minimum of the free energy of the 2d Yang-Mills theory at large-N*, *Mod. Phys. Lett.* **A 14** (1999) 751 [hep-th/9707080].

[72] M. Alimohammadi, M. Khorrami and A. Aghamohammadi, *Large-N limit of the generalized two-dimensional Yang-Mills theories*, *Nucl. Phys.* **B 510** (1998) 313 [hep-th/9707081].

[73] R. Dijkgraaf, R. Gopakumar, H. Ooguri and C. Vafa, *Baby universes in string theory*, hep-th/0504221.

[74] J. Hoppe, V. Kazakov and I.K. Kostov, *Dimensionally reduced SYM_4 as solvable matrix quantum mechanics*, *Nucl. Phys.* **B 571** (2000) 479 [hep-th/9907058].

[75] A. Abramowitz and I. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, chap. 17, Tenth Printing, December 1972, with corrections.

[76] R. Wong, *Asymptotic approximations of integrals*, Academic Press, New York 1989.

[77] X. Arsiwalla, R. Boels, M. Mariño and A. Sinkovics, *Phase transitions in q-deformed 2d Yang-Mills theory and topological strings*, hep-th/0509002.

[78] D. Jafferis and J. Marsano, *A dk phase transition in q-deformed Yang-Mills on S^2 and topological strings*, hep-th/0509004.